DIFFERENTIAL EQUATIONS, CLASS EXERCISE 12

(1) The current I in an LC series circuit is governed by the initial value problem
$$I''(t) + 4I(t) = g(t), \qquad I(0) = 0,\ I'(0) = 0,$$
where
$$g(t) = \begin{cases} 1 & 0 \le t < 1, \\ -1 & 1 \le t < 2, \\ 0 & 2 \le t. \end{cases}$$
Determine the current as a function of time t. Sketch the graph of the solution (use software).

(2) Laplace transform can also be used to solve systems of linear DE's. Consider the system
$$\begin{cases} x' - 2y &= 4t, & x(0) = 4 \\ y' + 2y - 4x &= -4t - 2, & y(0) = -5 \end{cases}$$
Let $X(s) = \mathcal{L}\{x\}$ and $Y(s) = \mathcal{L}\{y\}$. Taking Laplace transform of both sides of the system we obtain
$$\begin{cases} sX(s) - 4 - 2Y(s) &= \frac{4}{s^2} \\ sY(s) + 5 + 2Y(s) - 4X(s) &= -\frac{4}{s^2} - \frac{2}{s} \end{cases}$$
Solve this algebraic linear system and then implement inverse Laplace transform to obtain $x(t)$ and $y(t)$.

(3) Using integration by parts, show that
$$\int_{-\infty}^{\infty} f(t)\delta'(t)\ dt = -f'(0).$$
Next argue that
$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)\ dt = (-1)^n f^{(n)}(0).$$

(4) Solve the IVP
$$2y'' + y' + 2y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$
Sketch the graph of the solution (use software).

(5) In an LRC circuit with $L = 1H, R = 8\Omega$ and $C = 1/15F$, the capacitor initially carries a charge of $1C$ and no currents are flowing. There is no external voltage source. At time $t = 2s$, a power surge instantaneously applied an impulse voltage of $4\delta(t-2)$ into the system. Describe the charge of the capacitor over time. Sketch the graph (use software).