Diff Eq - Qex 2 - Solutions

(1) $y' - \frac{y}{3} = \sin t$, $\mu(t) = e^{-t/3}$; $\frac{d}{dt}\left[e^{-t/3} y\right] = e^{-t/3}\sin t$

$\int e^{-t/3}\sin t \, dt = -e^{-t/3}\cos t - \int (-1/3) e^{-t/3}(-\cos t)\, dt =$

$= -e^{-t/3}\cos t - \frac{1}{3}\int e^{-t/3}\cos t\, dt = -e^{-t/3}\cos t - \frac{1}{3}e^{-t/3}\sin t + \frac{1}{3}\int\left(-\frac{1}{3}\right)e^{-t/3}\sin t\,dt$

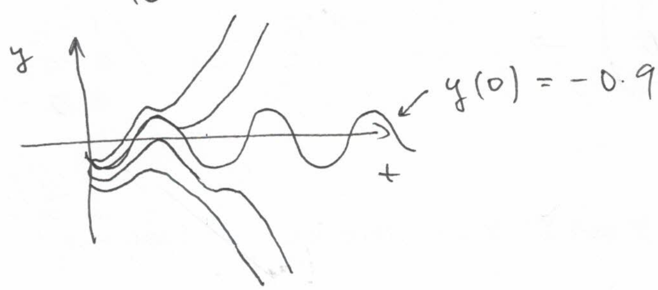
$\Rightarrow \int e^{-t/3}\sin t\, dt = -\frac{9}{10}e^{-t/3}\cos t - \frac{3}{10}e^{-t/3}\sin t + C$

$e^{-t/3} y = -\frac{9}{10}e^{-t/3}\cos t - \frac{3}{10}e^{-t/3}\sin t + C$

$y(t) = -\frac{9}{10}\cos t - \frac{3}{10}\sin t + Ce^{+t/3}$

$y(0) = -\frac{9}{10} + C \Rightarrow C = y(0) + \frac{9}{10}$

$y(t) = -\frac{9}{10}\cos t - \frac{3}{10}\sin t + \left(y(0) + \frac{9}{10}\right)e^{+t/3}$



(2) i) $\int \frac{dy}{y^3} = \int \frac{x\,dx}{(1+x^2)^{1/2}} \Rightarrow -\frac{1}{2y^2} = (1+x^2)^{1/2} + C$

$y(0) = 1 \Rightarrow -1/2 = 1 + C \Rightarrow C = -3/2$

$-\frac{1}{2y^2} = (1+x^2)^{1/2} - 3/2 \Rightarrow y^2 = \frac{1}{3 - 2(1+x^2)^{1/2}} \Rightarrow y = \pm\frac{1}{\sqrt{3-2(1+x^2)^{1/2}}}$

The initial condition selects the +.

$$y = \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

ii) The expression in the square root must be positive:

$3 - 2\sqrt{1+x^2} > 0 \Rightarrow 3/2 > \sqrt{1+x^2} \Rightarrow 9/4 > 1 + x^2 \Rightarrow 5/4 > x^2$

$\Rightarrow -\sqrt{5}/2 < x < \sqrt{5}/2$

(3) $\frac{dx}{dt} = 0.4(10-x)(5-x)$; $\frac{dx}{(10-x)(5-x)} = 0.4dt$; $\frac{1}{(10-x)(5-x)} = \frac{-1/5}{10-x} + \frac{1/5}{5-x}$

$-1/5\int\frac{dx}{10-x} + 1/5\int\frac{dx}{5-x} = \int 0.4dt$; $1/5 \ln|10-x| - 1/5\ln|5-x| = 0.4dt + C$

Diff Eq - Clex 2 - Solutions

$\ln\left|\frac{10-x}{5-x}\right| = 2t + C$; $\frac{10-x}{5-x} = ke^{2t}$; $t=0, x=0 \Rightarrow \frac{10-0}{5-0} = k \Rightarrow k=2$

$10 - x = 2e^{2t}(5-x)$; $x(t) = \frac{10-10e^{-2t}}{1-2e^{-2t}}$

$\lim_{t\to\infty} x(t) = 5$. It takes infinite time.

(4) $-\frac{dy}{y^3} = dx$; $\frac{1}{2y^2} = x + C$; $C = \frac{1}{2y_0^2} \Rightarrow y = \pm\left\{\frac{1}{2x + 1/y_0^2}\right\}^{1/2}$, and

take the + if $y_0 > 0$; take the − if $y_0 < 0$.

$2x + 1/y_0^2 > 0$, $x > -1/2y_0^2$. Interval of existence is $(-\frac{1}{2y_0^2}, \infty)$.

(5) $\frac{dN}{N(1-\frac{N}{100000})} = 0.3\,dt$, $\int\frac{dN}{N} + \int\frac{1/100000}{1-N/100000}\,dN = \int 0.3\,dt$

$\ln|N| - \ln\left|1 - \frac{N}{100000}\right| = 0.3t + C$, $\frac{N}{1-N/100000} = ke^{0.3t}$

$t=0, N=1 \Rightarrow k = \frac{100000}{99999}$

$N(t) = \frac{100000}{1+99999e^{-0.3t}}$; $95000 = \frac{100000}{1+99999e^{-0.3t}} \Rightarrow t = 48.19$ days.