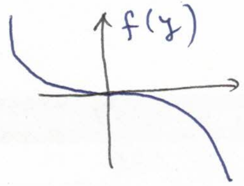
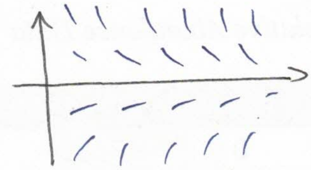
Diff Eq - Clex 3 - Solutions.

(1)

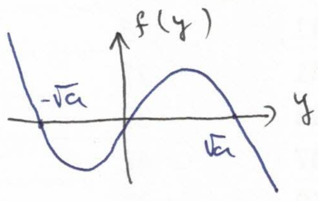
$a \le 0$

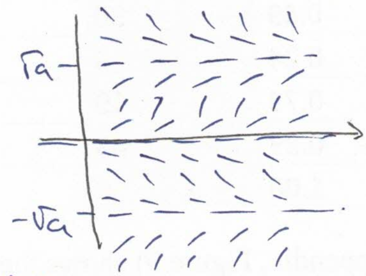




$y = 0$, stable

$a > 0$

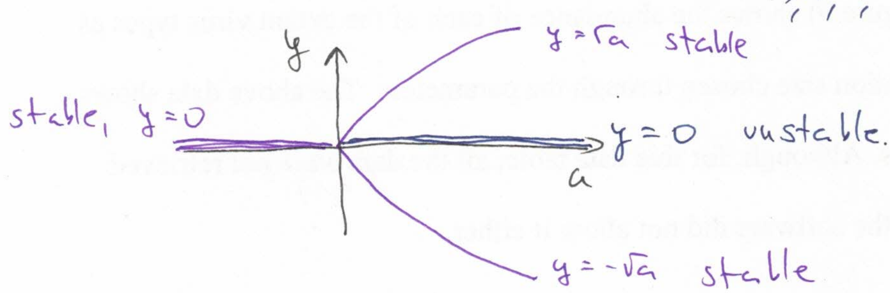




$y = \sqrt{a}$, stable

$y = 0$, unstable

$y = -\sqrt{a}$, stable



(2) $y' = -\beta y \Rightarrow y(t) = y_0 e^{-\beta t}$

$\frac{dx}{dt} = -\alpha x y = -\alpha x y_0 e^{-\beta t}$ $\qquad \frac{dx}{x} = -\alpha y_0 e^{-\beta t} dt$

$\ln|x| = -\alpha y_0 \left(\frac{e^{-\beta t}}{-\beta} + c \right)$ $\qquad x = C e^{\frac{\alpha y_0}{\beta} e^{-\beta t}}$

$x(0) = C e^{\alpha y_0/\beta} = x_0 \Rightarrow C = x_0 e^{-\alpha y_0/\beta}$

$x(t) = x_0 e^{\alpha y_0/\beta \left(e^{-\beta t} - 1 \right)}$

$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x_0 e^{\alpha y_0/\beta \left(e^{-\beta t} - 1 \right)} = x_0 e^{-\alpha y_0/\beta}$ escapes.

(3) $\underbrace{(ty^2 + \cos t)}_{M} dt + \underbrace{(e^{2y} + t^2 y)}_{N} dy = 0$ $\qquad \frac{\partial M}{\partial y} = 2yt = \frac{\partial N}{\partial t}$, exact.

$f(y,t) = \int (ty^2 + \cos t)\, dt = \frac{1}{2} t^2 y^2 + \sin t + g(y)$

$\frac{\partial f}{\partial y} = t^2 y + g'(y) = e^{2y} + t^2 y$; $\quad g(y) = \int e^{2y} dy = \frac{1}{2} e^{2y} + c$

$f(y,t) = \frac{t^2 y^2}{2} + \sin t + \frac{e^{2y}}{2} + c = 0$; $\quad t = \frac{\pi}{2}, y = 0 \Rightarrow c = -3/2$; $\quad \boxed{\frac{t^2 y^2}{2} + \sin t + \frac{e^{2y}}{2} = \frac{3}{2}}$

Diff Eq - Clex3 - Solutions

(4) $(x+2)\sin y\,dx + x\cos y\,dy = 0$

$$\frac{\partial M}{\partial y} = (x+2)\cos y \neq \frac{\partial N}{\partial x} = \cos y$$

Multiply by $\mu(x,y) = xe^x$

$$x(x+2)e^x \sin y\,dx + x^2 e^x \cos y\,dy = 0$$

$$\frac{\partial M}{\partial y} = x(x+2)e^x\cos y = \frac{\partial N}{\partial x} = (x^2+2x)e^x\cos y \quad \text{Exact.}$$

$$f(x,y) = \int x^2 e^x \cos y\,dy = x^2 e^x \sin y + g(x)$$

$$\frac{\partial f}{\partial x} = (x^2+2x)e^x\sin y + g'(x) = x(x+2)e^x \sin y \Rightarrow g'(x) = 0$$

$$f(x,y) = x^2 e^x \sin y = C$$

$$y = \arcsin\left(\frac{C}{x^2 e^x}\right).$$