DIFERENTIAL EQUATIONS, CLASS EXERCISE 3

(1) Consider the DE

$$\frac{dy}{dt} = y(a - y^2).$$

(i) Consider the three cases $a < 0, a = 0, a > 0$. In each case find the critical points and determine whether each critical point is asymptotically stable, semistable or unstable.

(ii) In each case sketch the directional field of the equation together with several integral curves.

(iii) Draw the bifurcation diagram of this DE, i.e. plot the location of the critical points versus the parameter a.

(2) Some diseases are spread largely by *carriers*, individuals who can transmit the disease but who exhibit no overt symptoms. Let x and y denote the proportions of susceptibles and carriers, respectively, in the population. Suppose that carriers are identified and removed from the population at rate β, so

$$\frac{dy}{dt} = -\beta y.$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$\frac{dx}{dt} = -\alpha xy.$$

a) Let $y(0) = y_0$ be the initial proportion of carriers. Determine the proportion of carriers as a function of time, i.e. find $y(t)$.

b) Use the result from part a) to find $x(t)$ subject to the initial condition $x(0) = x_0$.

c) Find the proportion of the population that escapes the epidemic by finding $\lim_{t\to\infty} x(t)$.

(3) Find an implicit solution for the initial value problem

$$(e^{2y} + t^2 y)y' + ty^2 + \cos t = 0, \quad y(\pi/2) = 0.$$

(4) Show that the equation below is not exact, but becomes exact when multiplied by the given integrating factor. Then solve the equation.

$$(x+2)\sin y + (x\cos y)y' = 0, \qquad \mu(x,y) = xe^{x}$$