DiffEq. - Clex 7 - Solutions

(1) $Lq'' + Rq' + \frac{1}{C}q = E(t)$; $0.2q'' + 300q' + 10^5 q = 0$

$$r_{1,2} = \frac{-300 \pm \sqrt{300^2 - 4(0.2)10^5}}{0.4}, \quad r_1 = -500, \quad r_2 = -1000$$

$$q(t) = c_1 e^{-500t} + c_2 e^{-1000t}$$

$10^{-6} = q(0) = c_1 + c_2$; $0 = I(0) = -500c_1 - 1000c_2 \Rightarrow c_1 = 2\times 10^{-6}$, $c_2 = -10^{-6}$

(2) Ansatz: $u(t) = A\cos\omega t + B\sin\omega t$; $u'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$

$u''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$; Plugging into the DE we have:

$\cos\omega t$: $-A\omega^2 + \frac{1}{4}B\omega + 2A = 2$

$\sin\omega t$: $-B\omega^2 - \frac{1}{4}A\omega + 2B = 0$

$$\Rightarrow A = \frac{2(2-\omega^2)}{(2-\omega^2)^2 + \omega^2/16}, \quad B = \frac{\omega/2}{(2-\omega^2)^2 + \omega^2/16}$$

$$R = \sqrt{A^2 + B^2} = \frac{2}{[(2-\omega^2)^2 + \omega^2/16]^{1/2}}$$

The max of R while varying ω happens when the denominator has a minimum.

$$[(2-\omega^2)^2 + \omega^2/16]' = 2(2-\omega^2)(-2\omega) + 1/8\,\omega = 0; \quad \omega\left(4\omega^2 - \frac{63}{8}\right) = 0$$

$$\omega_{max} = \sqrt{63/32} \approx 1.4; \quad R_{max} = \frac{2}{\left[\left(2-\frac{63}{32}\right)^2 + \frac{1}{16}\cdot\frac{63}{32}\right]^{1/2}} = \frac{64}{\sqrt{127}} \approx 5.679$$

(3) $u'' + 4u = 3\cos(1.8t)$

Particular solution: $u = A\cos(1.8t) + B\sin(1.8t)$

$$u'' = -3.24A\cos(1.8t) - 3.24B\sin(1.8t)$$

$\cos 1.8t$: $-3.24A + 4A = 3$ $\qquad A = \frac{75}{19} \approx 3.947$

$\sin 1.8t$: $-2.24B + 4B = 0$ $\qquad B = 0$

$$u(t) = c_1\cos 2t + c_2\sin 2t + \frac{75}{19}\cos 1.8t$$

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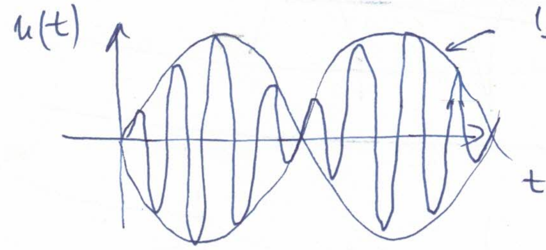
$$u(0) = 0 = c_1 + \frac{75}{19} \Rightarrow c_1 = -\frac{75}{19}$$

$$u'(0) = 0 = 2c_2 \Rightarrow c_2 = 0$$

$$u(t) = -\frac{75}{19}\cos 2t + \frac{75}{19}\cos 1.8t = \frac{75}{19}\left[\cos 1.8t - \cos 2t\right]$$

$$u(t) = \frac{150}{19}\sin(0.1t)\sin(1.9t)$$



$\frac{150}{19}\sin(0.1t)$ envelope.

(4)
$$m_1 \frac{d^2u_1}{dt^2} = -k_1u_1 + k_2(u_2-u_1) - f_1\frac{du_1}{dt}$$

$$m_2 \frac{d^2u_2}{dt^2} = -k_2(u_2-u_1) - k_3u_2 - f_2\frac{du_2}{dt}$$

New variables: $x_1 = u_1$, $x_2 = u_2$, $y_1 = u_1'$, $y_2 = u_2'$. The system reads:

$$x_1' = y_1$$

$$x_2' = y_2$$

$$y_1' = -\frac{f_1}{m_1}y_1 - \frac{(k_1+k_2)}{m_1}x_1 + \frac{k_2}{m_1}x_2$$

$$y_2' = -\frac{f_2}{m_2}y_2 + \frac{k_2}{m_2}x_1 - \frac{(k_2+k_3)}{m_2}x_2$$

In matricial form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & k_2/m_1 & -f_1/m_1 & 0 \\ k_2/m_2 & -\frac{(k_2+k_3)}{m_2} & 0 & -f_2/m_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$