Diff Eq. - Clex 8 - Solutions

1.

$$\det\begin{pmatrix} -2-\lambda & -2.5 \\ 10 & -2-\lambda \end{pmatrix} = \lambda^2+4\lambda+29 \; ; \quad \lambda_{1,2} = \frac{-4 \pm \sqrt{16-4\cdot 29}}{2} = -2 \pm 5i$$

$$\lambda_1 = -2+5i, \quad \left(\begin{array}{cc|c} -5i & -2.5 & 0 \\ 10 & -5i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -i/2 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad x = \begin{pmatrix} is/2 \\ s \end{pmatrix}, \quad x_1 = \begin{pmatrix} i \\ 2 \end{pmatrix}$$

$$\lambda_2 = -2-5i, \quad x_2 = \begin{pmatrix} -i \\ 2 \end{pmatrix}$$

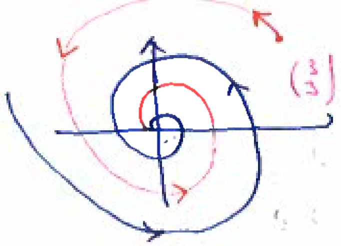
$$x_1 = e^{-2t+5it}\begin{pmatrix} i \\ 2 \end{pmatrix} = e^{-2t}(\cos 5t + i \sin 5t)\begin{pmatrix} i \\ 2 \end{pmatrix} =$$

$$= e^{-2t}\begin{pmatrix} -\sin 5t \\ 2\cos 5t \end{pmatrix} + i e^{-2t}\begin{pmatrix} \cos 5t \\ 2\sin 5t \end{pmatrix}$$

$$x(t) = c_1 e^{-2t}\begin{pmatrix} -\sin 5t \\ 2\cos 5t \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} \cos 5t \\ 2\sin 5t \end{pmatrix}$$

$$x(0) = c_1\begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \Rightarrow c_1 = 3/2, \; c_2 = 3$$

$$x(t) = \frac{3}{2} e^{-2t}\begin{pmatrix} -\sin 5t \\ 2\cos 5t \end{pmatrix} + 3e^{-2t}\begin{pmatrix} \cos 5t \\ 2\sin 5t \end{pmatrix}$$



2.

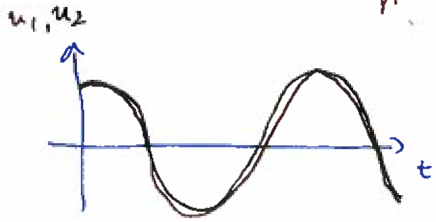
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{(k_2+k_3)}{m_2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 3 & 0 & 0 \\ 9/4 & -13/4 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

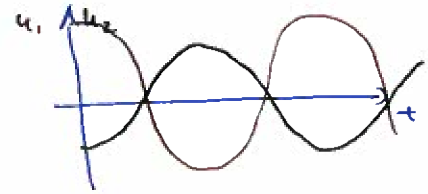
$$\lambda_1 = i, \; \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix}; \quad \lambda_2 = -i, \; \vec{\xi}_2 = \begin{pmatrix} 1 \\ 1 \\ -i \\ -i \end{pmatrix}; \quad \lambda_3 = 2.5i, \; \xi_3 = \begin{pmatrix} 4 \\ -3 \\ -10i \\ 7.5i \end{pmatrix}; \quad \lambda_4 = -2.5i, \; \xi_4 = \begin{pmatrix} 4 \\ -3 \\ 10i \\ -7.5i \end{pmatrix}$$

$$x^1(t) = e^{it}\begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix} = \begin{pmatrix} \cos t \\ \cos t \\ -\sin t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{pmatrix} \qquad x^3(t) = e^{2.5it}\begin{pmatrix} 4 \\ -3 \\ -10i \\ 7.5i \end{pmatrix} = \begin{pmatrix} 4\cos 2.5t \\ -3\cos 2.5t \\ 10\sin 2.5t \\ -7.5\sin 2.5t \end{pmatrix} + i\begin{pmatrix} 4\sin 2.5t \\ -3\sin 2.5t \\ -10\cos 2.5t \\ 7.5\sin 2.5 \end{pmatrix}$$

$$\vec{x}(t) = c_1\begin{pmatrix} \cos t \\ \cos t \\ -\sin t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{pmatrix} + c_3\begin{pmatrix} 4\cos 2.5t \\ -3\cos 2.5t \\ 10\sin 2.5t \\ -7.5\sin 2.5t \end{pmatrix} + c_4\begin{pmatrix} 4\sin 2.5t \\ -3\sin 2.5t \\ -10\cos 2.5t \\ 7.5\sin 2.5t \end{pmatrix}$$

In mode 1 the two masses move in synchronous fashion and the two velocities are $\pi/2$ ahead. In mode 2 the two masses move in synchron; the velocities are $\pi/2$ behind. In mode 3 the two masses move against each other and the velocities are $\pi/2$ behind. In mode 4 the two masses move against each other and the velocities are $\pi/2$ ahead.





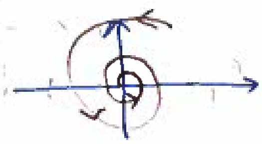
3 $\det\begin{pmatrix} 3-\lambda & \alpha \\ -6 & -4-\alpha \end{pmatrix} = \lambda^2 + \lambda - 12 + 6\alpha$. $\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{49-24\alpha}$

critical value: $49 - 24\alpha = 0$, $\alpha = 49/24$

critical value: $49 - 24\alpha = 1$, $\alpha = 2$

Case 1: $\alpha > 49/24$

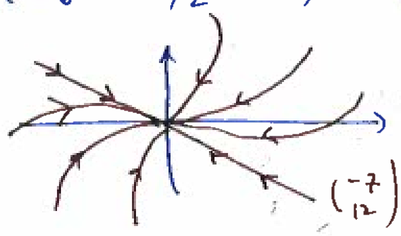
$\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{24\alpha - 49}\, i$



Spiral.
Asymptotically stable.

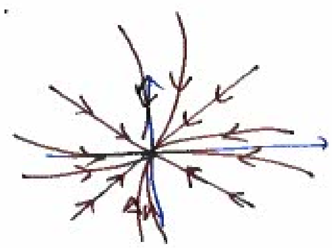
Case 2: $49 - 24\alpha = 0 \Rightarrow \alpha = 49/24$, $\lambda_{1,2} = -1/2$

$\left(\begin{array}{cc|c} 7/2 & 49/24 & 0 \\ -6 & -7/2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 7/12 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\vec{\xi}_1 = \begin{pmatrix} -7 \\ 12 \end{pmatrix}$, missing second eigenvector



Improper node.
Asymptotically stable.

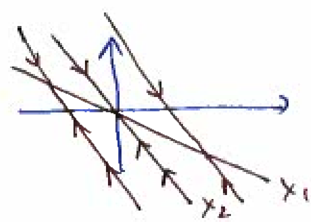
case 3: $2 < \alpha < \frac{49}{24}$



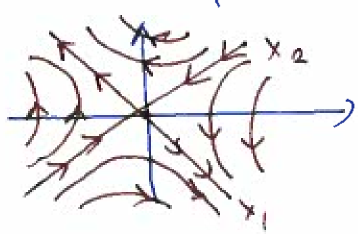
Two distinct, real eigenvalues, $\lambda_1 < 0$, $\lambda_2 < 0$

case 4: $\alpha = 2$

$\lambda_1 = 0$ $x_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\lambda_2 = -1$ $x_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



case 5: $\alpha < 2$, $\lambda_1 > 0$, $\lambda_2 < 0$



Saddle point.

Diff Eq - Clex 8 - Solutions

(4) $\det\begin{pmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{pmatrix} = \lambda^2+\lambda = 0$, $\lambda_1 = 0$, $\lambda_2 = -1$

$\lambda_1 = 0$ $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right) \Rightarrow \xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -1$ $\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right) \Rightarrow \xi_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Let $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 = 2$, $c_2 = -1$

$$x^{(1)}(t) = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Let $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow c_1 = -1$, $c_2 = 1$

$$x^{(2)}(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Phi(t) = \left(x^{(1)}(t), x^{(2)}(t)\right) = \begin{pmatrix} 2-e^{-t} & -1+e^{-t} \\ 2-2e^{-t} & -1+2e^{-t} \end{pmatrix}$$

$$x(t) = \Phi(t)\, x(0) = \Phi(t)\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5+4e^{-t} \\ -5+8e^{-t} \end{pmatrix}$$