Diff Eq - §1 - DE Modelling; Direction Fields

Time flow: "St. Augustine - "What then is time? If no one asks me I know what it is? If I wish to explain it to him who asks, I do not know"."

observation: We can explain natural behaviour using relations involving rates at which thing happen. In mathematical terms:

relations = equations; rates of change = derivatives

⇒ Differential Equations.

Def: A differential equation that describes some natural process is a **mathematical model** for the process.

Ex: Propagation of waves (sound, EM); stock market behaviour (stochastic); changes in probability distribution in QM (Schrödinger equation).

Ex: Object falling: t - time, v - velocity (vertical)

Newton's second law: $F = ma = m\frac{dv}{dt}$

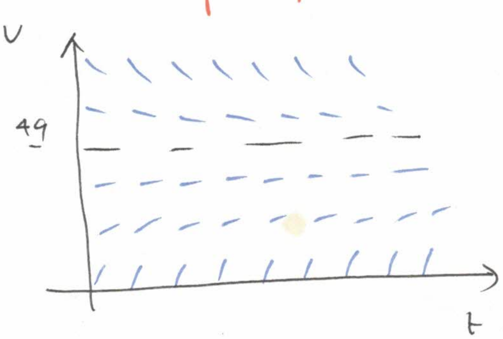
$F = mg - \gamma v$ ← gravity and drag force. $\uparrow \gamma v$ $\downarrow mg$

$$m\frac{dv}{dt} = mg - \gamma v \quad \leftarrow \text{the mathematical model.}$$

Say $m = 10\,kg$, $\gamma = 2\,kg/s$; $g = 9.8\,m/s^2$ ⇒ $10\frac{dv}{dt} = 10\cdot(9.8) - 2v$

$$\boxed{\frac{dv}{dt} = 9.8 - 0.2v}$$

We will solve this equation later, but for the moment let's plot the direction (slope) field.



$v=0$ $\frac{dv}{dt} = 9.8$; $v = 20$ $\frac{dv}{dt} = 5.8$

$v = 40$ $\frac{dv}{dt} = 1.8$; $v = 49$ $\frac{dv}{dt} = 0$

$v = 60$ $\frac{dv}{dt} = -2.2$

Diff. Eq. - §1 - DE Modelling: Direction Fields

$v = v(t)$ has a graph which is a curve in the tv-plane. Each line segment on the direction field is tangent to one of the solution curves.

Notice that $v(t) = 49$ is a solution of our differential equation, since it is constant it is called equilibrium solution (perfect balance between gravity and drag. If $v < 49$ the direction field shows $\frac{dv}{dt} > 0$; if $v > 49$ $\frac{dv}{dt} < 0$; all solutions are converging on $v = 49$ → terminal velocity.

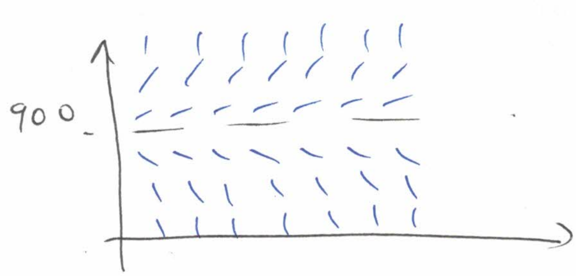
rem: More generally a direction field can be draw (and are quite useful) for differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

ex: Show the DF's for $y' = y^2 - 2y$ and $y' = e^{-t} - 2y$ on the Web Page.

ex: Population numbers p for field mice; with hawks as predators.

$$\frac{dp}{dt} = 0.5p - 450 \qquad t\text{-months}$$



Here the equilibrium solution $p = 900$ is a repeller.

Q: ex: In general for the equation $\frac{dy}{dt} = ky - e$ where is the equilibrium solution? Is it an attractor or a repeller?

ex: A drug is administered at a rate 300 cm3/h. Concentration of the drug is 3mg/cm3. The drug leaves the bloodstream at a rate proportional to the amount present with rate constant $0.2h^{-1}$. Write the diff. eq. $\frac{dM}{dt} = 900 - 0.2M$