Diff. Eq. - §10 - 2nd order DE

Def: A second order ordinary DE has the form
$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \qquad (*)$$
It is linear if $f(t,y,y') = g(t) - p(t)y' - q(t)y$ and then we write the equation as
$$y'' + p(t)y' + q(t)y = g(t). \qquad (**)$$
→ as before numerical and geometrical approaches are more appropriate in the nonlinear case.

Def: An initial value problem for $(*)$ now has a pair of initial conditions: $y(t_0) = y_0$, $y'(t_0) = y'_0$. (So we specify the initial point and the initial slope).

Def: A second order linear DE $(**)$ is homogeneous if $g(t) = 0$, i.e.
$$y'' + p(t)y' + q(t)y = 0$$
Otherwise it is nonhomogeneous.

We will start with the simplest homogeneous case: the coefficients are constant:
$$ay'' + by' + cy = 0 \qquad a,b,c = \text{const.}$$
Ex: $y'' - y = 0$, $y(0) = 1$, $y'(0) = -3$.

Solution is: $y(t) = c_1e^t + c_2e^{-t}$

$c_1 + c_2 = 1$, $c_1 - c_2 = -3$ → $c_1 = -1$ $c_2 = 2$
$$y(t) = -e^t + 2e^{-t}.$$
Ex: $ay'' + by' + cy = 0$. Let's try the ansatz $y = e^{rt}$.
$$(ar^2 + br + c)e^{rt} = 0$$
$$ar^2 + br + c = 0 \quad \leftarrow \text{characteristic equation.}$$

Say the characteristic equation has two real roots r_1, r_2. Then

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$ is a solution since (general solution)

$$ay'' + by' + cy = c_1(ar_1^2 + br_1 + c)e^{r_1 t} + c_2(ar_2^2 + br_2 + c)e^{r_2 t} = 0.$$

Given the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$ we can determine the constants c_1 and c_2:

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0, \quad c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'$$

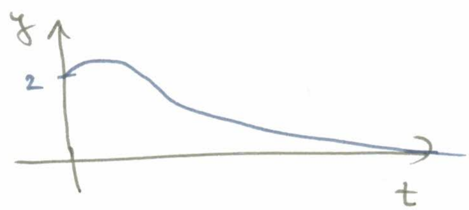
$$\Rightarrow c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \quad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

Ex: Solve the initial value problem $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$

$r^2 + 5r + 6 = 0$, $r_1 = -2$, $r_2 = -3$ $\quad y = c_1 e^{-2t} + c_2 e^{-3t}$

$c_1 + c_2 = 2$, $-2c_1 - 3c_2 = 3$ $\Rightarrow$ $c_1 = 9$, $c_2 = -7$

$y = 9e^{-2t} - 7e^{-3t}$



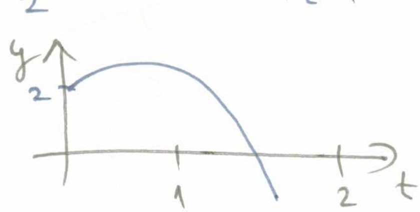
Ex: Solve the initial value problem

$4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = 1/2$

$4r^2 - 8r + 3 = 0$ $\quad r_1 = 3/2$, $r_2 = 1/2$ $\quad y = c_1 e^{3t/2} + c_2 e^{t/2}$

$c_1 + c_2 = 2$, $\frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}$ $\Rightarrow$ $c_1 = -1/2$, $c_2 = 5/2$

$y = -1/2\, e^{3t/2} + 5/2\, e^{t/2}$



Where is the max? $\quad y' = -3/4\, e^{3t/2} + 5/4\, e^{t/2} = 0$

$\frac{3}{4} e^{3t/2} = 5/4\, e^{t/2}$, $e^t = 5/3$, $t_m = \ln 5/3 = 0.511$, $y_m = 2.152$