Diff Eq - §11 - Solutions of linear homogeneous 2nd order D.equations

Goal: Built a picture of the solutions of a 2nd order linear homogeneous DE

$$y'' + p(t)y + q(t)y = 0$$

Th: (Existence and Uniqueness).

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

where p, q, g are continuous on an open interval I s.t. $t_0 \in I$. There exists a unique solution to this problem, and the solution exists throughout the interval I. ☒ (The existence proof will be given in the constructive techniques ahead of us).

Ex: Which is the longest interval on which the solution for

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0, \quad y(1) = 2, y'(1) = 1 \quad \text{exists.}$$

$$p(t) = \frac{t}{t^2 - 3t} = \frac{1}{t-3}, \quad q(t) = \frac{1}{t} \to I = (0, 3).$$

Th: If y_1 and y_2 are two solutions of the DE

superposition

$$y'' + p(t)y' + q(t)y = 0$$

Then so is the linear combination $c_1y_1 + c_2y_2$ for any value of the constants c_1 and c_2.

Pr: $(c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) =$

$= c_1(y_1'' + py_1' + q) + c_2(y_2'' + py_2' + q) = 0$ ☒

Def: The Wronskian of the solutions y_1, y_2 is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Ex: $y_1 = e^{-2t}, \ y_2 = e^{-3t} \Rightarrow W(y_1, y_2) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$

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Th: Suppose that $y_1(t)$ and $y_2(t)$ are two solutions of

$$y'' + p(t)y' + q(t)y = 0,$$

and that the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$ are assigned.

Then it is possible (always) to choose constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

solves the initial value problem iff

$$W(y_1, y_2)(t_0) \neq 0.$$

Pr:
$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$ ▨

Th: Suppose that y_1 and y_2 are two solutions of the DE $y'' + p(t)y' + q(t)y = 0$. Then the family of solutions $y = c_1 y_1(t) + c_2 y_2(t)$ includes every solution of this equation iff $\exists$ a point t_0 where $W(y_1, y_2)(t_0) \neq 0$.

Pr: $\Leftarrow$ Let $\phi(t)$ be any solution of the DE. We must express it as

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t) \quad \text{for some } c_1, c_2.$$

Let t_0 be a point where $W(y_1, y_2)(t_0) \neq 0$. With $y_0 = \phi(t_0)$, $y_0' = \phi'(t_0)$, $\phi(t)$ solves the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

But by the previous Th we can choose c_1, c_2 so that $y(t)$ is also a solution of the initial value problem. By uniqueness of the solution

$$\phi(t) = y(t) = c_1 y_1(t) + c_2 y(t).$$

$\Rightarrow$ Now suppose the Wronskian is identically zero. Then there are values

y_0, y_0' so that the initial conditions for $y = c_1y_1 + c_2y$:

$$c_1y_1(t_0) + c_2y_2(t_0) = 0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = 0$$

cannot be solved. But by the existence and uniqueness Th we have a solution of this initial value problem, say $\phi(t)$. Then $\phi(t)$ is not of the form $c_1y_1 + c_2y_2$. ☒

Cor: To find all solutions of $y'' + p(t)y' + q(t)y = 0$ we just need to find two solutions so that their Wronskian is not identically zero, say y_1, y_2. In this case y_1, y_2 is called fundamental set of solutions and

$$y = c_1y_1 + c_2y_2$$

the general solution.

Ex: Let $y_1(t) = e^{r_1t}$ and $y_2(t) = e^{r_2t}$, $r_1 \neq r_2$ be two solutions of $y'' + p(t)y' + q(t)y = 0$. They form a fundamental set.

Pr: $W(y_1, y_2) = \begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r_2t} \end{vmatrix} = (r_2 - r_1)e^{(r_1+r_2)t}$ ☒

Ex: $y_1(t) = \sqrt{t}$ and $y_2(t) = 1/t$ form a fundamental set for

$$2t^2y'' + 3ty' - y = 0$$

Pr: Check that they are solutions.

$$W = \begin{vmatrix} \sqrt{t} & 1/t \\ \frac{1}{2}\frac{1}{\sqrt{t}} & -1/t^2 \end{vmatrix} = -\frac{3}{2}t^{-3/2} \neq 0$$

Prop: + $y'' + p(t)y' + q(t) = 0$.

Let y_1 be the solution for the initial conditions $y(t_0) = 1$, $y'(t_0) = 0$.
Let y_2 — " — $y(t_0) = 0$, $y'(t_0) = 1$. Then y_1, y_2 is a fundamental set.

Pr: $W(y_1, y_0)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$ ☒

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Ex: $y''-y=0$. General solution is $y=c_1e^t+c_2e^{-t}$

$y(0)=1,\ y'(0)=0,\quad c_1=c_2=1/2,\quad y_1=\frac{1}{2}(e^t+e^{-t})=\cosh t$

$y(0)=0,\ y'(0)=1,\quad c_1=1/2,\ c_2=-1/2\quad y_2=\frac{1}{2}(e^t-e^{-t})=\sinh t$

$$W(y_1,y_2)=\begin{vmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{vmatrix}=\cosh^2 t-\sinh^2 t=1.$$

Thus $\{\cosh t, \sinh t\}$ is a fundamental set; but so is $\{e^t, e^{-t}\}$.

Th: (Abel's Theorem). Let y_1, y_2 be solutions of $y''+p(t)y'+q(t)y=0$ where p, q are continuous on an open interval I. Then

$$W(y_1,y_2)(t)=c\exp\left[-\int p(t)\,dt\right],\quad c=\text{const}\,(y_1,y_2).$$

Further $W(y_1,y_2)(t)=0\ \forall t\in I\ (c=0)$ or $W(y_1,y_2)(t)\neq 0\ \forall t\in I$.

Pr: $y_1''+p(t)y_1'+q(t)y_1=0 \quad *(-y_2)$
$y_2''+p(t)y_2'+q(t)y_2=0 \quad *(y_1)$ $\Big\}+$

$$y_1y_2''-y_1''y_2+p(t)(y_1y_2'-y_1'y_2)=0 \quad (*)$$

But $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=y_1y_2'-y_1'y_2$; $W'=\cancel{y_1'y_2'}+y_1y_2''-y_1''y_2\cancel{-y_1'y_2'}$

$(*)\Rightarrow W'+p(t)W=0$

$\Rightarrow W(t)=c\,e^{-\int p(t)dt}$. ∎

rem: Under the conditions of the Th the Wronskian is never zero or identically zero.

Ex: $y_1(t)=\sqrt{t},\ y_2(t)=1/t$ are solutions of $2t^2y''+3ty'-y=0$ with Wronskian $W(y_1,y_2)(t)=-\frac{3}{2}t^{-3/2}$. Now

$$p(t)=\frac{3t}{2t^2}=\frac{3}{2t},\quad W(y_1,y_2)(t)=c\,e^{-\left[\int\frac{3}{2t}dt\right]}=ce^{-\frac{3}{2}\ln t}=ct^{-3/2}. \ ∎$$

Summary: To find the general solution of $y''+p(t)y'+q(t)y=0$ on $\alpha<t<\beta$.

i) Find two solutions y_1, y_2 s.t. $W(y_1,y_2)\neq 0$ on $\alpha<t<\beta$

ii) The general solution is $y=c_1y_1(t)+c_2y_2(t)$

iii) If initial conditions are specified, we can always determine c_1, c_2. ∎

$\alpha<t_0<\beta$