Diff. Eq - §12 - Complex Roots of The Characteristic Equation

rem: The equation $ay'' + by' + cy = 0$, $a,b,c \in \mathbb{R}$ has solutions of the type $y = e^{rt}$ when $ar^2 + br + c = 0$

If $b^2 - 4ac < 0$. The roots of the equation are

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu, \quad \lambda, \mu \in \mathbb{R}.$$

and the solutions are

$$y_1(t) = e^{(\lambda + i\mu)t}, \quad y_2(t) = e^{(\lambda - i\mu)t}.$$

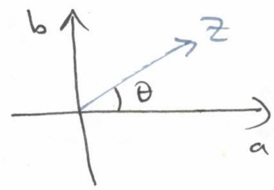
Scholium: Complex numbers.

$(2+3i)+(1-i) = ?$ $\qquad (3-i)(2+4i) = ?$

$\frac{3+2i}{1-i} = ?$ $\qquad$ Solve $x^2 + 4 = 0$. $\qquad$ Solve $2x^2 + x + 2 = 0$.

$z = a + ib$

$a = \mathrm{Re}\, z \quad b = \mathrm{Im}\, z$



$|z| = \sqrt{a^2 + b^2}$ (absolute value)

$\theta = \arctan y/x$ (argument).

Find $|z|, \theta$ for $z = 1+3i$, $z = -2 - 0.5i$, $z = -i$, $z = -1$.

Functions of complex numbers:

ex: $z = 2+i$, $p(z) = z^2 - z + 1 = ?$

What is meant by e^z? Well $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, $-\infty < t < \infty$

So then $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

In particular: $e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{k=0}^{\infty} \frac{(i)^{2k} t^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i)^{2k+1} t^{2k+1}}{(2k+1)!}$

$= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = \cos t + i \sin t$

Euler formula: $\boxed{e^{it} = \cos t + i \sin t}$

ex: $e^{i\pi} = -1$, $e^{i\pi/2} = i$, $e^{4+2i} = e^4(\cos(2) + i\sin(2))$.

Diff. Eq. - §12 - Complex Roots of the characteristic equation

Ex: Solve the initial value problem

$$y'' + y' + \frac{37}{4} y = 0 \qquad y(0) = 2, \quad y'(0) = 8.$$

Sol: $r^2 + r + 37/4 = 0 \qquad y_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 37/4}}{2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i$

General solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$, where

$$y_1(t) = e^{(-1/2+3i)t} = e^{-t/2}(\cos 3t + i \sin 3t)$$

$$y_2(t) = e^{(-1/2-3i)t} = e^{-t/2}(\cos 3t - i \sin 3t).$$

We can check: $W(y_1, y_2) = \begin{vmatrix} e^{(-1/2+3i)t} & e^{(-1/2-3i)t} \\ (-1/2+3i)e^{(-1/2+3i)t} & (-1/2-3i)e^{(-1/2-3i)t} \end{vmatrix} = -6ie^{-t} \neq 0$

solving the initial conditions will require solving

$$2 = y(0) = c_1 + c_2 \; ; \quad 8 = y'(0) = c_1(-1/2 + 3i) + c_2(-1/2 - 3i).$$

However observe that

$$u(t) = e^{-t/2}\cos 3t, \quad v(t) = e^{-t/2} \sin 3t$$

is also a fundamental set of solutions.

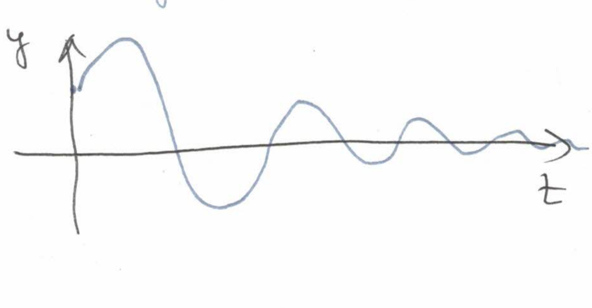
$$W(u,v) = \begin{vmatrix} e^{-t/2}\cos 3t & e^{-t/2}\sin 3t \\ -1/2\, e^{-t/2}\cos 3t - 3e^{-t/2}\sin 3t & -1/2\, e^{-t/2}\sin 3t + 3e^{-t/2}\cos 3t \end{vmatrix} = 3e^{-t} \neq 0.$$

So any solution can be written as

$$y = c_1 u(t) + c_2 v(t) = c_1 e^{-t/2}\cos 3t + c_2 e^{-t/2} \sin 3t$$

Now: $2 = y(0) = c_1$, $8 = y'(0) = -1/2\, c_1 + 3c_2 \Rightarrow c_2 = 3 \Rightarrow$

$$y(t) = e^{-t/2}(2\cos 3t + 3 \sin 3t).$$



Diff. Eq - §12 - Complex roots.

Complex roots the general case: $ay'' + by' + cy = 0$. Say the characteristic equation has roots $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$. A fundamental set of real-valued solutions is

$$u(t) = e^{\lambda t}\cos \mu t, \quad v(t) = e^{\lambda t}\sin \mu t$$

In fact $W(u,v) = \mu e^{2\lambda t}$ (Note $\mu \neq 0$ since the roots are complex)

The general solution is:

$$\boxed{y = c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t}$$

Ex: $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$.

Sol: $16r^2 - 8r + 145 = 0$, $r_{1,2} = 1/4 \pm 3i$

$y = c_1 e^{t/4}\cos 3t + c_2 e^{t/4}\sin 3t$

$-2 = y(0) = c_1$; $1 = y'(0) = \frac{1}{4}c_1 + 3c_2 \rightarrow c_2 = 1/2$

$y(t) = -2e^{t/4}\cos 3t + 1/2\, e^{t/4}\sin 3t$

Ex: $y'' + 4y = 0$ $y(0) = 1$, $y'(0) = -2$

$r^2 + 4 = 0$, $r = \pm 2i$

$y_1 = c_1 \cos 2t$, $y_2 = c_2 \sin 2t$, $y = c_1\cos 2t + c_2 \sin 2t$

$1 = y(0) = c_1$; $-2 = y'(0) = 2c_2$ $c_2 = -1$

$y(t) = \cos 2t - \sin 2t$