Diff. Eq. - §13 - Repeated roots. Reduction of order.

Case III: $ay'' + by' + cy = 0 \to ar^2 + br + c = 0$ and $b^2 - 4ac = 0$.

Now $r_1 = r_2 = -b/2a$ and $y_1(t) = e^{-bt/2a}$ but we are missing a second solution.

Ex: Solve $y'' + 6y' + 9y = 0 \to (r+3)^2 = 0$ $\quad r_{1,2} = -3$, $y_1(t) = e^{-3t}$

Basic idea: $y_2(t) = v(t)\,y_1(t) = v(t)\,e^{-3t}$

$$(v''e^{-3t} - 6v'e^{-3t} + 9ve^{-3t}) + 6(v'e^{-3t} - 3ve^{-3t}) + 9ve^{-3t} = 0.$$

$$(v'' - 6v' + 6v' + 9v - 18v + 9v)e^{-3t} = 0 \Rightarrow v''(t) = 0 \Rightarrow$$

$v(t) = k_1 t + k_2$ $\qquad y_1(t) = e^{-3t}$

$y(t) = c_1 e^{-3t} + (k_1 t + k_2)e^{-3t} = c_1 e^{-3t} + c_2 t e^{-3t}$ $\qquad y_2(t) = te^{-3t}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t} - 3te^{-3t} \end{vmatrix} = e^{-6t} \neq 0.$$

In general: $ay'' + by' + cy = 0$, case $b^2 - 4ac = 0$ i.e. $r_1 = r_2 = -b/2a$

The general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-r_1 t} + c_2 t e^{-r_1 t}, \quad r_1 = -b/2a$$

$W(y_1, y_2) = e^{-bt/a} \neq 0.$

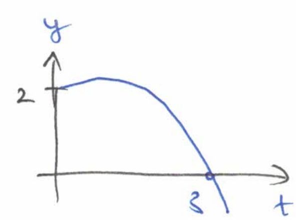
Ex: Solve the initial value problem

$$y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}$$

$r^2 - r + 1/4 = 0$ $\quad r_{1,2} = 1/2$, $y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$

$y(0) = c_1 = 2$; $y'(0) = \frac{1}{2}c_1 + c_2 = 1/3 \Rightarrow c_2 = -2/3$

$y(t) = 2e^{t/2} - \frac{2}{3}te^{t/2}$



Summary: 2nd order linear homogeneous DE's with constant coefficients

$ay'' + by' + cy = 0$ $\qquad$ Let r_1, r_2 be the roots of $ar^2 + br + c = 0$

Case 1: $r_1, r_2 \in \mathbb{R}$; $r_1 \neq r_2$

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Diff. Eq - §13 - Repeated roots. Reduction of order

Case 2: $r_{1,2} = \lambda \pm i\mu \in \mathbb{C}$

$$y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t = d_1 e^{r_1 t} + d_2 e^{r_2 t}$$

Case 3: $r_1 = r_2 = r \in \mathbb{R}$

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_2 t}$$

Reduction of Order: Suppose we know one solution $y_1(t) \neq 0$ of

$$y'' + p(t)y' + q(t)y = 0.$$

To find a second solution let $y = v(t)y_1(t)$. Then

$$y' = v'y_1 + vy_1' \;:\; y'' = v''y_1 + 2v'y_1' + vy_1''$$

substituting we have

$$y_1 v'' + (2y_1' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{\to 0}v = 0$$

$\Rightarrow y_1 v'' + (2y_1' + py_1)v' = 0$ which is a 1st order linear DE for v'

ex: Given that $y_1(t) = 1/t$ is a solution of $2t^2y'' + 3ty' - y = 0$ find a fundamental set of solutions.

$$y = v(t)/t \;;\; y' = v't^{-1} - vt^{-2} \;;\; y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

$$\Rightarrow 2tv'' + (-4+3)v' + \underbrace{(4t^{-1} - 3t^{-1} - t^{-1})}_{\to 0}v = 2tv'' - v' = 0$$

Let $v' = w \Rightarrow 2tw' - w = 0 \qquad \frac{dw}{w} = \frac{1}{2}\frac{dt}{t} \Rightarrow w(t) = ct^{1/2} \Rightarrow$

$$\Rightarrow v(t) = \tfrac{2}{3}ct^{3/2} + k \Rightarrow y(t) = v(t)/t = \tfrac{2}{3}ct^{1/2} + k/t \Rightarrow y_2(t) = \sqrt{t}$$

$$W(y_1, y_2) = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \tfrac{1}{2}t^{-1/2} \end{vmatrix} = \tfrac{3}{2}t^{-3/2} \neq 0. \quad \boxtimes$$

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