Diff Eq. - §14 - Method of Undetermined coefficients

Consider the nonhomogeneous equation $y''+p(t)y'+q(t)y=g(t)$. The equation $y''+p(t)y'+q(t)y=0$ is the associated homogeneous equation.

Prop. If Y_1 and Y_2 are two solutions of the nonhomogeneous equation

$$y''+p(t)y'+q(t)y=g(t),$$

then Y_1-Y_2 is a solution of the associated homogeneous equation.

Pr: $(Y_1-Y_2)''+p(t)(Y_1-Y_2)'+q(t)(Y_1-Y_2)=[Y_1''+p(t)Y_1'+q(t)Y_1]-[Y_2''+p(t)Y_2'+q(t)Y_2]=g(t)-g(t)=0.$ ☒

Th: The general solution of $y''+p(t)y'+q(t)y=g(t)$ can be written in the form

$$y(t)=c_1y_1(t)+c_2y_2(t)+Y(t),$$

where y_1, y_2 are a fundamental set of solutions of the corresponding homogeneous equation, and Y is some specific solution of the nonhomogeneous equation.

Pr: By the previous Prop. $Y_1-Y_2=c_1y_1(t)+c_2y_2(t)$. ☒

rem: so to solve the nonhomogeneous DE we must solve the homogeneous DE and find a single solution of the nonhomogeneous one.

Here we will focus on finding a particular solution $Y(t)$ of the nonhomogeneous equation.

Method of undetermined coefficients: works best when the homogeneous equation has constant coefficients and the nonhomogeneous term is restricted to a small class of functions: polynomials, exp, trig.

Despite these restrictions the method has many practical implications.

Ex: Find a particular solution of $y''-3y'-4y=3e^{2t}$

Sol: Consider $Y(t)=ce^{2t}$. Then $Y'(t)=2ce^{2t}$, $Y''(t)=4ce^{2t}$

$(4c-6c-4c)e^{2t}=3e^{2t}$, $c=-1/2 \Rightarrow Y(t)=-1/2\, e^{3t}$

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Ex: Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t$$

Sol: Since $\sin t$ becomes $\cos t$ under differentiation we start with the ansatz: $Y(t) = A\sin t + B\cos t$

$$Y'(t) = A\cos t - B\sin t, \quad Y''(t) = -A\sin t - B\cos t$$

substituting we have

$$(-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t$$

$$-5A + 3B = 2 \qquad -3A - 5B = 0 \quad \Rightarrow \quad A = -5/17 \quad B = 3/17$$

$$Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Ex: $y'' - 3y' - 4y = 4t^2 - 1$

Sol: Ansatz $Y(t) = At^2 + Bt + C$; $Y'(t) = 2At + B$; $Y''(t) = 2A$

$$2A - 6At - 3B - 4At^2 - 4Bt - 4C = 4t^2 - 1$$

$$(-4A)t^2 + (-6A - 4B)t + (2A - 3B - 4C) = 4t^2 - 1$$

$$A = -1 \quad B = 3/2 \quad C = -15/8$$

Ex: $y'' - 3y' - 4y = -8e^t\cos 2t$

Sol: Ansatz $Y(t) = Ae^t\cos 2t + Be^t\sin 2t$

$$Y'(t) = (A + 2B)e^t\cos 2t + (-2A + B)e^t\sin 2t$$

$$Y''(t) = (-3A + 4B)e^t\cos 2t + (-4A - 3B)e^t\sin 2t$$

substituting we see that: $10A + 2B = 8$, $2A - 10B = 0 \Rightarrow A = \frac{10}{13}, B = \frac{2}{13}$

$$Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t.$$

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rem: If $g(t) = g_1(t) + g_2(t)$ then solve

$$ay'' + by' + cy = g_1(t) \rightarrow Y_1(t)$$

$$ay'' + by' + cy = g_2(t) \rightarrow Y_2(t)$$

and $Y(t) = Y_1(t) + Y_2(t)$.

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Ex: $y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t$

By splitting the RHS we have

$y'' - 3y' - 4y = 3e^{2t} \rightarrow Y_1(t) = -\frac{1}{2} e^{2t}$

$y'' - 3y' - 4y = 2\sin t \rightarrow Y_2(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$

$y'' - 3y' - 4y = -8e^t\cos 2t \rightarrow Y_3(t) = \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t.$

$Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t \sin 2t.$ $\boxtimes$

Ex: $y'' - 3y' - 4y = 2e^{-t}$

Ansatz: $Y(t) = Ae^{-t}$. Then $Y'(t) = -Ae^{-t}$; $Y''(t) = Ae^{-t}$

Substituting: $Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t} \rightarrow 0 = 2e^{-t}!$

This happened because the nonhomogeneous term $g(t) = 2e^{-t}$ is a solution of the homogeneous equation.

New ansatz: $Y(t) = Ate^{-t}$. Then

$Y'(t) = Ae^{-t} - Ate^{-t}$; $Y''(t) = -2Ae^{-t} + Ate^{-t}$

substituting we have

$(-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t} \Rightarrow A = -\frac{2}{5}$

$Y(t) = -\frac{2}{5} te^{-t}.$