Diff. Eq. - §15 - Variation of parameters

rem: Let's look at the 1st order case first.

$$\frac{dy}{dt} + p(t)y = g(t)$$

The associated homogeneous equation is $\frac{dy}{dt} + p(t)y = 0$ with solution $y(t) = e^{-\int p(t)dt} = 1/\mu(t)$. Then the solution of (

is: $Y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right] = y(t)\,[\underbrace{\int \mu(t)g(t) + c}_{u(t)}]$

$Y(t) = u(t)\,y(t)$.

↑ variation of parameters.

constr: $+ y'' + p(t)y' + q(t)y = g(t)$ (1)

Say we have the general solution of the corresponding homogeneous equation:

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

crucial idea: replace the constants c_1 and c_2 with functions $u_1(t)$ and $u_2(t)$.

$$y(t) = u_1(t)y_2(t) + u_2(t)y_2(t)$$

Now: $y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$. Set the terms involving $u_1'(t)$ and $u_2'(t)$ to zero:

$$\boxed{u_1'y_1 + u_2'y_2 = 0.} \quad (2)$$

y' simplifies to: $y' = u_1y_1' + u_2y_2'$. Differentiate again

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

substitute y, y', y'' in equation (1) and reorder terms

$$u_1[y_1'' + p(t)y_1' + q(t)y_1] + u_2[y_2'' + p(t)y_2' + q(t)y_2] +$$

$$+ u_1'y_1' + u_2'y_2' = g(t)$$

$$\Rightarrow \boxed{u_1' y_1' + u_2' y_2' = g(t)} \quad (3).$$

Using Cramer's rule to solve the linear (algebraic) system of equations for u_1' and u_2' we have:

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}, \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}.$$

Finally:

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt + c_1 \; ; \; u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt + c_2$$

Th: If the functions $p(t)$, $q(t)$ and $g(t)$ are continuous on an open interval I, and if the functions $y_1(t)$, $y_2(t)$ are a fundamental set of solutions of the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ corresponding to the equation

$$y'' + p(t)y' + q(t)y = g(t) \quad (1)$$

then a particular solution of (1) is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}ds$$

for $t_0 \in I$. The general solution of (1) is:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t). \;\square$$

Ex: Find a particular solution of $y'' - 4y' + 4y = (t+1)e^{2t}$

Sol: $r^2 - 4r + 4 = 0 \qquad r_1 = r_2 = r = 2$

$$y_1(t) = e^{2t}, \quad y_2(t) = te^{2t} \qquad W(y_1,y_2)(t) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t}+e^{2t} \end{vmatrix} = e^{4t}$$

$$u_1' = -\frac{(t+1)te^{4t}}{e^{4t}} = -t^2 - t \; ; \; u_2' = \frac{(t+1)e^{4t}}{e^{4t}} = t+1$$

$$u_1 = -\frac{t^3}{3} - \frac{t^2}{2}, \quad u_2 = \frac{t^2}{2} + t$$

$$\Rightarrow Y(t) = \left(-\frac{1}{3}t^3 - \frac{1}{2}t^2\right)e^{2t} + \left(\frac{1}{2}t^2 + t\right)te^{2t} = \frac{1}{6}t^3e^{2t} + \frac{1}{2}t^2e^{2t}$$

$$y(t) = c_1e^{2t} + c_2te^{2t} + \frac{1}{6}t^3e^{2t} + \frac{1}{2}t^2e^{2t} \leftarrow \text{general solution.}$$

Ex: Solve $4y'' + 36y = \csc 3t$

$y'' + 9y = \frac{1}{4}\csc 3t$, $r^2 + 9 = 0$ $r_{1,2} = \pm 3i$

$y_1(t) = \cos 3t$, $y_2(t) = \sin 3t$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3$$

$$u_1' = -\frac{\sin 3t \cdot 1/4 \csc 3t}{3} = -\frac{1}{12} \; ; \; u_2' = \frac{\cos 3t \cdot 1/4 \csc 3t}{3} = \frac{1}{12}\cot 3t$$

$$u_1 = -\frac{1}{12}t \; ; \quad u_2 = \frac{1}{36}\ln|\sin 3t|$$

$$y(t) = c_1\cos 3t + c_2\sin 3t - \frac{1}{12}t\cos 3t + \frac{1}{36}\sin(3t)\ln|\sin 3t|.$$

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