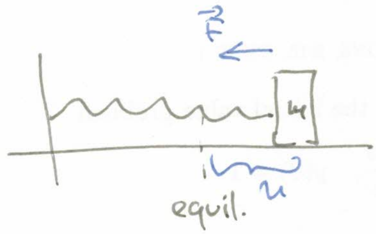
Diff. Eq. - §16 - Mechanical & Electrical Vibrations

Context:



Hooke's Law: $F = -ku$

u - displacement from equilibrium

But then $ma = mu'' = F$

$mu'' = -ku$; $mu'' + ku = 0$, $u = u(t)$.

If there is also friction $F_{fr} = -\gamma u'(t)$ (viscous resistance); and there might be an external force $F(t)$.

$$ma = mu'' = -ku - \gamma u'(t) + F(t)$$

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

$m, \gamma, k > 0$; m - mass; γ - damping constant; k - spring const.

We are also given initial position and initial velocity:

$$u(0) = u_0, \quad u'(0) = V_0.$$

Undamped Free Vibrations. Suppose $F(t) = 0$; $\gamma = 0$.

$$mu'' + ku = 0 \Rightarrow mr^2 + k = 0 \Rightarrow r = \pm i\sqrt{k/m}$$

Let $\omega_0 = \sqrt{k/m}$. The general solution is

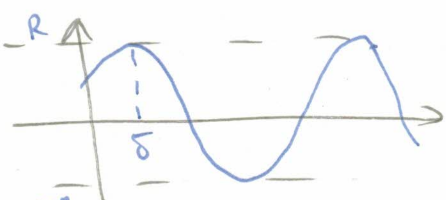
$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad (1)$$

It is convenient to rewrite eq. (1) as

$$u(t) = R\cos(\omega_0 t - \delta) \quad \text{or}$$

$$u(t) = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t.$$

Thus $A = R\cos\delta$, $B = R\sin\delta$; $R = \sqrt{A^2 + B^2}$, $\tan\delta = B/A$



The period of the motion is:

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$; ω_0 - natural frequenc[y]

R - amplitude ; δ - phase.

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Ex. A 10 kg mass is attached to a spring with $k = 6N/cm$. The mass is displaced 10 cm from equilibrium and let go with no initial push. Solve the equations of motion.

$m = 10 \quad k = 600 \quad \omega_0 = \sqrt{k/m} = \sqrt{60} \approx 7.75$

$u(t) = A \cos\sqrt{60}\, t + B \sin\sqrt{60}\, t$

$u'(t) = -A\sqrt{60} \sin\sqrt{60}\, t + B\sqrt{60} \cos\sqrt{60}\, t$

$u(0) = 0.1 = A \quad ; \quad 0 = u'(0) = B\sqrt{60} \Rightarrow B = 0$

$u(t) = 0.1 \cos\sqrt{60}\, t.$

$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{10}{600}} = 0.811\,s. \quad ; R = 0.1m \; ; \delta = 0$

Damped Free Vibrations.

$$m u'' + \gamma u' + k u = 0 \Rightarrow m r^2 + \gamma r + k = 0$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right).$$

Case 1: $\gamma^2 > 4km$, $u(t) = Ae^{r_1 t} + Be^{r_2 t}$ $\quad r_1, r_2 < 0$ (1)

overdamped

Case 2: $\gamma^2 = 4km$; $u(t) = (A + Bt)\, e^{-\gamma t/2m}$. (2)

critically damped

Case 3: $\gamma^2 < 4km$; $u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t)$ (3)

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m} > 0.$$

Notice that in all cases $\lim_{t\to\infty} u(t) = 0$.

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rem: The most important case is the 3rd one in which the damping is small. Let again $A = R\cos\delta$ and $B = R\sin\delta$, then (3) becomes

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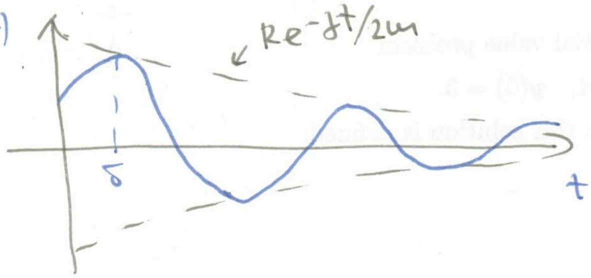
$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta).$$



← Damped vibration.

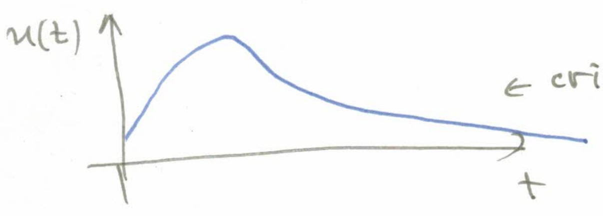
The motion is not truly periodic; μ - quasi-frequency, and $T_d = 2\pi/\mu$ - quasi-period. Comparing T_d to the period T of the undamped motion we have:

$$\frac{T_d}{T} = \frac{2\pi/\mu}{2\pi/\omega_0} = \frac{\omega_0}{\mu} = \frac{\sqrt{k/m}}{\left(\sqrt{4km-\gamma^2}\right)/2m} = \left(1-\frac{\gamma^2}{4km}\right)^{-1/2} \cong \left(1+\frac{\gamma^2}{8km}\right).$$

When $\frac{\gamma^2}{4km}$ is small, then damping has a small effect on the "quasi-period".

As $\gamma^2 \to 4km$, $\mu \to 0$ and $T_d \to \infty$.

At $\gamma^2 = 4km$ the motion is critically damped: $u(t) = (A+Bt)e^{-\gamma t/2m}$



← critically damped.

At $\gamma^2 > 4km$ the motion is overdamped. $u(t) = Ae^{r_1 t} + Be^{r_2 t}$, $r_1, r_2 < 0$

Ex: The motion of a certain spring-mass system is governed by

$$u'' + \frac{1}{8}u' + u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

Solve for $u(t)$. Find the quasi-period.

Sol: $r^2 + \frac{1}{8}r + 1 = 0 \qquad r_{1,2} = \dfrac{-1/8 \pm \sqrt{1/64 - 4}}{2} = -\frac{1}{16} \pm \frac{\sqrt{255}}{16}i$

$$u(t) = e^{-t/16}\left[A\cos\frac{\sqrt{255}}{16}t + B\sin\frac{\sqrt{255}}{16}t\right]$$

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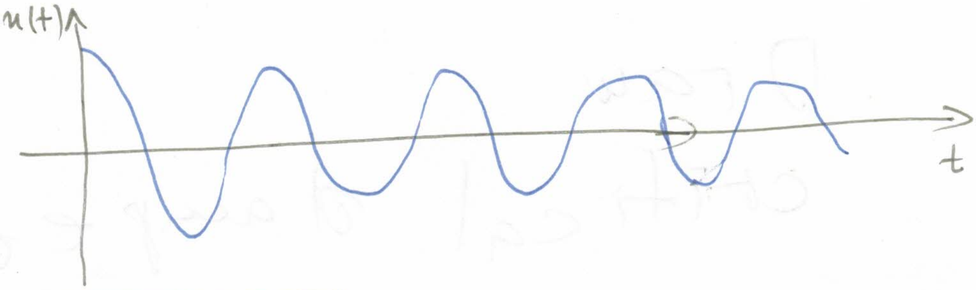
$2 = u(0) = A$; $0 = u'(0) = -\frac{1}{16}A + \frac{\sqrt{255}}{16}B$, $B = \frac{2}{\sqrt{255}}$

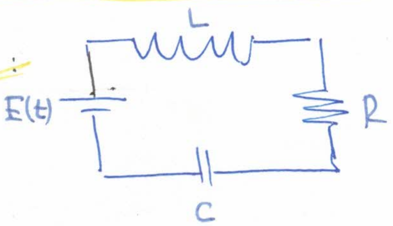
$u(t) = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) =$

$= \frac{32}{\sqrt{255}}\, e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$; $\delta = \arctan\frac{1}{\sqrt{255}} \approx 0.06$

$\mu_d = \sqrt{255}/16$, $T_d = 2\pi/\mu \approx 6.3$ (sec.).



Ex:



The voltage drops across the inductor, resistor and capacitor add up to Kirchhoff's law

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t) \quad , \; i = \frac{dq}{dt}$$

$$\Rightarrow \boxed{L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).}$$

Charact. equation is $Lr^2 + Rr + 1/C = 0$

$R^2 - 4L/C > 0$ - overdamped ; $R^2 - 4L/C = 0$ - critically damped

$R^2 - 4L/C < 0$ - underdamped.

Ex: Find the charge $q(t)$ on the capacitor in an LRC circuit with $L = 0.25\,H$, $R = 10\,\Omega$, $C = 0.001\,F$, $E(t) = 0$, $q(0) = 6C$, $i(0) = 0\,A$.

Sol: $\frac{1}{4}q'' + 10q' + 1000q = 0 \rightarrow q'' + 40q' + 4000q = 0$

$$r_{1,2} = \frac{-40 \pm \sqrt{1600 - 16000}}{2} = -20 \pm 60i$$

$q(t) = c_1 e^{-20t}\cos 60t + c_2 e^{-20t}\sin 60t$; $6 = q(0) = c_1$; $0 = q'(0) = -20c_1 + 60$

$q(t) = 6e^{-20t}\cos 60t + 2e^{-20t}\sin 60t = 2\sqrt{10}\, e^{-20t}\cos(60t - 0.322)$.