Diff. Eq. – §17 – Forced Vibrations

Ex: $u'' + u' + \frac{5}{4}u = 3\cos t$, $u(0) = 2$, $u'(0) = 3$

Sol: A particular solution is sought of the form $Y(t) = A\cos t + B\sin t$

$Y' = -A\sin t + B\cos t$, $Y'' = -A\cos t - B\sin t$

$(-A\cos t - B\sin t) + (-A\sin t + B\cos t) + \frac{5}{4}(A\cos t + B\sin t) = 3\cos t$.

$\sin t: -B - A + \frac{5}{4}B = 0$, $\cos t: -A + B + \frac{5}{4}A = 3$

$\Rightarrow A = \frac{12}{17}, B = \frac{48}{17} \Rightarrow Y(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t$

Next we solve the homogeneous system: $r^2 + r + \frac{5}{4} = 0$, $r_{1,2} = -\frac{1}{2} \pm i$

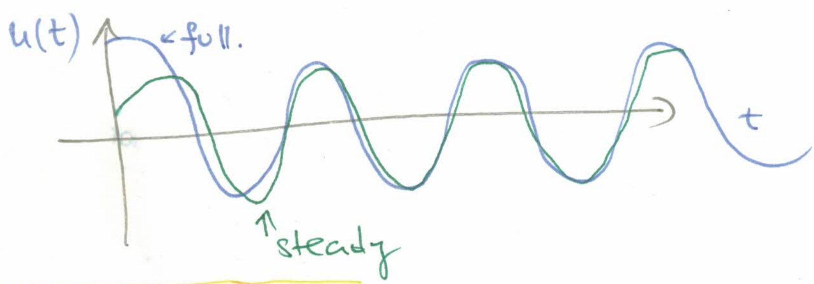
$Y_h(t) = c_1 e^{-1/2 t}\cos t + c_2 e^{-1/2 t}\sin t$

$\Rightarrow u(t) = c_1 e^{-1/2 t}\cos t + c_2 e^{-1/2 t}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t$

$2 = u(0) = c_1 + \frac{12}{17}$; $3 = u'(0) = -\frac{1}{2}c_1 + c_2 + \frac{48}{17}$

$\Rightarrow c_1 = \frac{22}{17}$, $c_2 = \frac{14}{17}$ $\Rightarrow$

$$u(t) = \underbrace{\frac{22}{17} e^{-1/2 t}\cos t + \frac{14}{17} e^{-1/2 t}\sin t}_{\text{transient terms}} + \underbrace{\frac{12}{17}\cos t + \frac{48}{17}\sin t}_{\text{steady state}}$$



General spring-mass system subject to external force $F(t)$.

$$m u''(t) + \gamma u'(t) + k u(t) = F(t)$$

Suppose $F(t) = F_0 \cos \omega t$. As in the example above the solution is

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{\substack{\text{transient solution,}\\ \text{has negative exponent,}\\ \text{dies out.}}} + \underbrace{A\cos\omega t + B\sin\omega t}_{\substack{\text{steady state = forced response.}\\ U(t)}}$$

- The transient solution enables us to satisfy the initial conditions
- The damping term dissipates the effect of the initial conditions

Diff. Eq. - §17 - Forced Vibrations

Let's write the steady state in the form

$$U(t) = A\cos\omega t + B\sin\omega t = R\cos(\omega t - \delta).$$

It could be shown that

$$R = \frac{F_0}{\Delta}, \quad \tan\delta = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}, \text{ where}$$

$$\Delta = \sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}, \quad \omega_0^2 = k/m$$

Q: How does the amplitude R of the steady state oscillation depend on the frequency ω of the external force?

F_0/k - the static displacement of the spring under force F_0.

$$\frac{R}{F_0/k} = \frac{\cancel{F_0}\, k}{\cancel{F_0}\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} = \frac{1}{\sqrt{\frac{m^2}{k^2}(\omega_0^2-\omega^2)^2 + \frac{\gamma^2}{k^2}\omega^2}}$$

$$\frac{R}{F_0/k} = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}}}, \quad \Gamma = \frac{\gamma^2}{mk}$$

As $\omega \to 0$ $R \to F_0/k$; As $\omega \to \infty$, $R \to 0$. The max of R occurs when $\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}$ has a min, i.e.

$$2\left(1-\frac{\omega^2}{\omega_0^2}\right)\left(\frac{-2\omega}{\omega_0^2}\right) + \Gamma\frac{2\omega}{\omega_0^2} = 0 \Rightarrow, \quad \omega = 0 \text{ or}$$

$$\Rightarrow \omega_{max}^2 = \omega_0^2\left(1-\frac{\Gamma}{2}\right) = \omega_0^2\left(1-\frac{\gamma^2}{2mk}\right) = \omega_0^2 - \frac{\gamma^2}{2m^2} \quad (1)$$

The maximum value of R is

$$R_{max} = \frac{F_0}{k\sqrt{\left(1-\left(1-\frac{\Gamma}{2}\right)\right)^2 + \Gamma\left(1-\Gamma/2\right)}} = \frac{F_0}{k\sqrt{\Gamma - \Gamma^2/4}} = \frac{F_0}{k\sqrt{\Gamma}\sqrt{1-\Gamma/4}}$$

$$R_{max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \quad (2)$$

When $\gamma^2/mk > 2$, then ω_{max} given by eq. (1) is imaginary. In this case the max value of R occurs for $\omega = 0$.

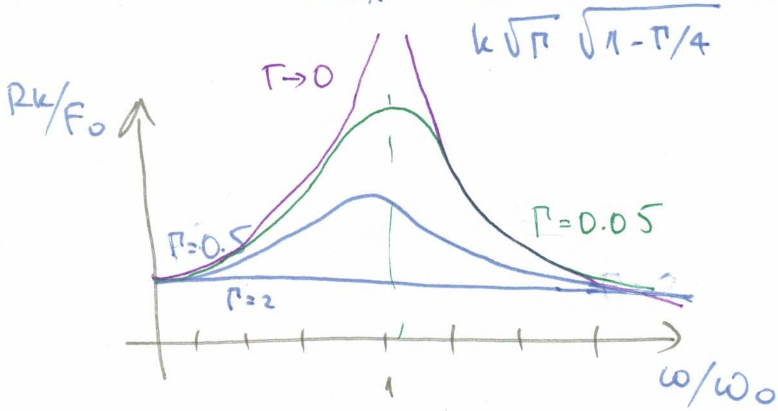
Recall that the critical damping occurs when $\frac{\gamma^2}{mk} = 4$.

Thus eq. 2 is only meaningful for lightly damped systems $\gamma^2 < 2mk$. So consider such lightly damped systems. For small γ

$$\omega_{max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} \approx \omega_0^2$$

So when the external driving force is at a frequency close to the natural frequency ω_0 we have max amplitude of the forced response. This max amplitude is

$$R_{max} = \frac{F_0}{k\sqrt{\Gamma}\sqrt{1-\Gamma/4}} = \frac{F_0}{\gamma\omega_0\sqrt{1-\gamma^2/4mk}}, \quad \Gamma < 2.$$



Resonance → The amplitude of the forced response is large even for small external forces when γ is small and $\omega \sim \omega_0$.

→ resonance should be taken seriously in the design of structures (or electrical circuits)

→ resonance is useful for design of instruments → seismographs

The phase also depends on ω in an interesting way.

$$\sin\delta = \frac{\gamma\omega}{\sqrt{m(\omega_0^2-\omega^2) + \gamma^2\omega^2}}$$

when $\omega \sim 0$, $\sin\delta \sim 0$, thus $\delta \sim 0$. The response is nearly in phase with the external force.

When $\omega = \omega_0$, $\sin\delta = 1 \Rightarrow \delta \sim \pi/2$.

When $\omega \to \infty$, $\sin\delta \sim 0$ but $\delta \sim \pi$, so the response is nearly out of phase with the excitation.

Ex: $u'' + \frac{1}{8}u' + u = 3\cos\omega t$, $u(0)=2$, $u'(0)=0$.

$\omega_0 = \sqrt{k/m} = 1$, $\Gamma = \frac{\gamma^2}{km} = 1/64$

Let's solve the initial value problem at resonance $\omega = \omega_0 = 1$.

Homog.: $r^2 + \frac{1}{8}r + 1 = 0$ $\qquad r_{1,2} = \frac{-1/8 \pm \sqrt{1/64 - 4}}{2} = -\frac{1}{16} \pm \frac{\sqrt{255}\,i}{16}$

$y_h(t) = c_1 e^{-1/16 t}\cos\frac{\sqrt{255}}{16}t + c_2 e^{-1/16 t}\sin\frac{\sqrt{255}}{16}t$

Ansatz for a particular solution $Y(t) = A\cos t + B\sin t$

$Y' = -A\sin t + B\cos t$; $Y'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t - \frac{1}{8}A\sin t + \frac{1}{8}B\cos t + A\cos t + B\sin t = 3\cos t$

$\cos t$: $-A + \frac{1}{8}B + A = 3$ $\qquad B = 24$

$\sin t$: $-B - \frac{1}{8}A + B = 0$ $\qquad A = 0$

$Y(t) = 24\sin t$

8×3 $\qquad$ phase lag $\pi/2$

$y(t) = c_1 e^{-1/16 t}\cos\frac{\sqrt{255}}{16}t + c_2 e^{-1/16 t}\sin\frac{\sqrt{255}}{16}t + 24\sin t$

$2 = y(0) = c_1$, $\quad 0 = -\frac{c_1}{16} + c_2\frac{\sqrt{255}}{16} + 24$ $\qquad c_2 = -\frac{382}{\sqrt{255}} = -23.92$

$y(t) = 2e^{-1/16 t}\cos\frac{\sqrt{255}}{16}t - 23.92\, e^{-1/16 t}\sin\frac{\sqrt{255}}{16}t + 24\sin t$

Forced vibrations without damping. $\gamma = 0$

Diff. eq. - §17 - Forced Vibrations

$mu'' + ku = F_0 \cos\omega t$ $\qquad mr^2 + k = 0 \qquad r_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$

$u_{homog.} = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t.$

Ansatz for the particular solution: $u = A\cos\omega t + B\sin\omega t$

$u' = -A\omega\sin\omega t + B\omega\cos\omega t\,; \quad u'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

$\cos\omega t: \; -mA\omega^2 + kA = F_0 \qquad \sin\omega t: \; -mB\omega^2 + kB = 0$

$$\Rightarrow B = 0\,, \quad A = \frac{F_0}{k - m\omega^2} = \frac{F_0}{m(\omega_0^2 - \omega^2)} \qquad \omega \neq \omega_0$$

$$u(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t$$

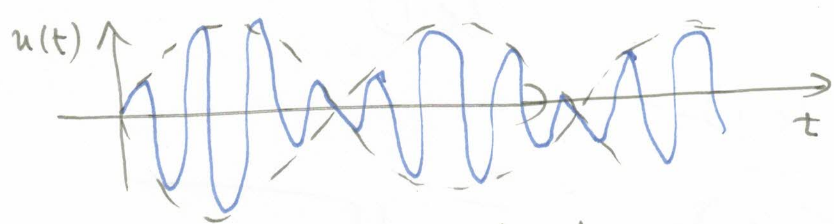
Let's specify initial rest: $u(0) = 0\,,\; u'(0) = 0 \Rightarrow$

$$C_1 = -\frac{F_0}{m(\omega_0^2-\omega^2)}\,, \quad C_2 = 0$$

$$\Rightarrow u(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}(\cos\omega t - \cos\omega_0 t)$$

$$u(t) = \frac{2F_0}{m(\omega_0^2-\omega^2)} \underbrace{\sin\frac{(\omega_0-\omega)t}{2}}_{\text{slow motion}} \; \underbrace{\sin\frac{(\omega_0+\omega)t}{2}}_{\text{Fast oscilation.}}$$

When $\omega \sim \omega_0 \Rightarrow$



We have a beat with slowly varying amplitude

$$\frac{2F_0}{m|\omega_0^2-\omega^2|}\left|\sin\frac{(\omega_0-\omega)}{2}t\right|.$$

(This also goes under the name amplitude modulation)

Ex. $u'' + u = \frac{1}{2}\cos 0.8t\,, \quad u(0) = 0\,, \; u'(0) = 0$

$\omega_0 = 1,\; \omega = 0.8\,, \quad u(t) = \frac{25}{9}\sin(0.1t)(\sin 0.9t)$ Picture above.

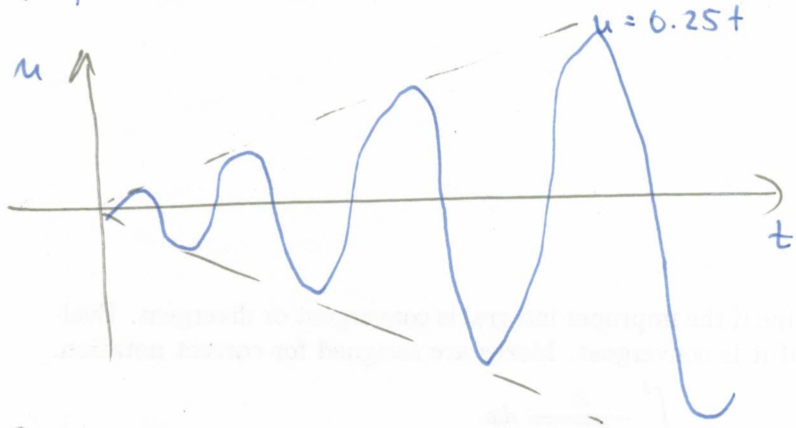
Q: What if $\omega = \omega_0$. Then the unhomogeneous term $F_0 \cos \omega t$ is a solution of the homogeneous equation $mu'' + ku = 0$. The general solution then is:

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

Ex.: $u'' + u = \frac{1}{2} \cos t$, $u(0) = 0$, $u'(0) = 0$

$u(t) = c_1 \cos t + c_2 \sin t + 0.25 t \sin t$ $\qquad c_1 = c_2 = 0$

$u(t) = 0.25 t \sin t$



- The motion becomes unbounded (and unhinged).

Example of a 3rd order DE §17.9

Ex: $y''' + y'' = e^x \cos x$

A particular solution would be of the form $y_p = Ae^x \cos x + Be^x \sin x$

$$y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x$$

$$\Rightarrow -2A + 4B = 1, \quad -4A - 2B = 0 \quad \Rightarrow A = -1/10, \; B = 1/5$$

$$y_p(x) = -\frac{1}{10} e^x \cos x + \frac{1}{5} e^x \sin x$$

For the solution of the homogeneous equation : $r^3 + r^2 = 0 \Rightarrow$

$$r_1 = r_2 = 0, \quad r_3 = -1$$

$$y_h(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-x} = c_1 + c_2 x + c_3 e^{-x}$$

$$\Rightarrow y(x) = c_1 + c_2 x + c_3 e^{-x} - \frac{1}{10} e^x \cos x + \frac{1}{5} e^x \sin x.$$

The coefficients c_1, c_2, c_3 could be determined by specifying initial conditions $y(0) = y_0$, $y'(0) = y_0'$, $y''(0) = y_0''$.

Caseu:
- homogeneous equation
- characteristic equation (constant coefficients).
- fundamental set of solutions → Wronskian

e.g. $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

- method of undetermined coefficients.
- method of variation of parameters.