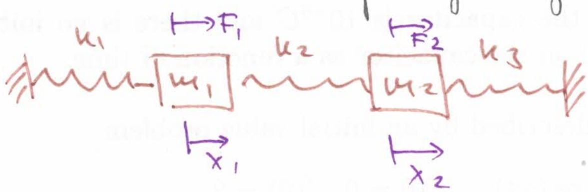
Diff. Eq. - §18 - Systems of linear equations. An Introduction.

context: The dynamics of several distinct elements is linked.

Moreover a DE of higher-order (or system of equations of higher-order) can be written as a system of 1st order equations.

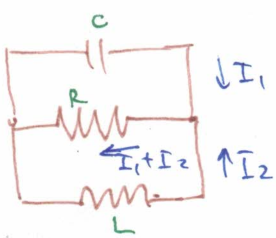
Ex: Consider the mass-springs system:



$$m_1 \frac{d^2x_1}{dt^2} = k_2(x_2 - x_1) - k_1x_1 + F_1(t) = -(k_1+k_2)x_1 + k_2x_2 + F_1(t)$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_3x_2 - k_2(x_2 - x_1) + F_2(t) = k_2x_1 - (k_2+k_3)x_2 + F_2(t).$$

Ex:



$$\frac{1}{C}Q_1 + R(I_1+I_2) = 0 \quad \to \quad \frac{1}{C}I_1 + R\left(\frac{dI_1}{dt} + \frac{dI_2}{dt}\right) = 0.$$

$$R(I_1+I_2) + L\frac{dI_2}{dt} = 0$$

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rem: When numerical approach is planed it is usefull to rewrite a higher order DE (system of higher-order DE's) as a system of 1st order DE's.

Ex: $u'' + \frac{1}{8}u' + u = 0$ let $x_1 = u$, $x_2 = u'$. The DE becomes

$$x_1' = x_2$$
$$x_2' = -x_1 - 1/8\, x_2.$$

Ex: The general one spring - one mass system $mu'' + \gamma u' + ku = F(t)$

becomes:
$$x_1' = x_2$$
$$x_2' = -k/m\, x_1 - \gamma/m\, x_2 + F(t)/m$$

$x_1 = u$
$x_2 = u'$

Ex: In general n'th order equation $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$ becomes

$$x_1' = x_2,\ x_2' = x_3, \dots, x_{n-1}' = x_n,\ x_n' = F(t, x_1, x_2, \dots, x_n)$$

where $x_1 = y$, $x_2 = y'$, ..., $x_n = y^{(n-1)}$

Diff Eq. - § - Systems of Linear Equations
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Thus systems of the form

$$\left.\begin{aligned} x_1' &= F_1(t, x_1, \dots, x_n) \\ &\ \vdots \\ x_n' &= F_n(t, x_1, \dots, x_n) \end{aligned}\right\} \quad (1)$$

include almost all cases of interest. These come (typically) with initial conditions: $x_1(t_0) = x_1^0, \dots, x_n(t_0) = x_n^0$. The solution of (1) $x_1(t), \dots, x_n(t)$ is a curve in n-dimensional space.

Th: Let each of the functions $F_1, \dots, F_n$ and the partial derivatives $\frac{\partial F_1}{\partial x_1}, \dots, \frac{\partial F_1}{\partial x_n}, \dots, \frac{\partial F_n}{\partial x_1}, \dots, \frac{\partial F_n}{\partial x_n}$ be continuous in a open region R of the $t, x_1, \dots, x_n$-space which contains the initial point. Then there is an interval $|t - t_0| < h$ in which there exists a unique solution to (1) that satisfies the initial conditions.

rem: Note that h could be small.

Def: The system (1) is linear if it is of the form:

$$\begin{aligned} x_1' &= p_{11}(t)\, x_1 + \dots + p_{1n}(t)\, x_n + g_1(t) \\ x_2' &= p_{21}(t)\, x_1 + \dots + p_{2n}(t)\, x_n + g_2(t) \\ &\ \vdots \\ x_n' &= p_{n1}(t)\, x_1 + \dots + p_{nn}(t)\, x_n + g_n(t) \end{aligned} \quad (2)$$

If $g_1(t) = \dots = g_n(t) = 0$ the system of DE's (2) is homogeneous.

rem: The mechanical/electrical examples above a linear!

Th: If the functions $p_{11}, p_{12}, \dots, p_{nn}, g_1, \dots, g_n$ are continuous on an open interval I there exists a unique solution of the system (2), that also satisfies the initial conditions $x_1(t_0) = x_1^0, \dots, x_n(t_0) = x_n^0$, where $t_0 \in I$. The solution exists throughout the interval I (unlike the nonlinear case). $\boxtimes$

rem: Notice (2) can be written as $\vec{x}\,' = P(t)\, \vec{x} + \vec{g}(t)$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{x}\,' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}, \quad g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$