Diff. Eq. - §19 - Eigenvalues and Eigenvectors

rem: For a matrix A the equation $AX = Y$ can be viewed as a linear transformation transforming the vector X into a vector Y. We are interested in directions that a preserved under the matrix action: $AX = \lambda X$, λ - scalar.

Def: If for a $n \times n$ matrix A we have $AX = \lambda X$, X - $n\times 1$ vector, $\lambda \in \mathbb{C}$, then λ is called an eigenvalue of A and X is called an eigenvector of A.

Ex: $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ $\quad X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad AX = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \lambda = 5.$

constr: How do we find eigenvalues & eigenvectors systematically?

$$AX = \lambda X, \quad AX - \lambda I X = 0, \quad (A - \lambda I)X = 0$$

For nontrivial solutions we must have

$$\det(A - \lambda I) = 0 \quad \leftarrow \text{characteristic equation}$$

Ex: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{pmatrix} = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2)$$

$\lambda_1 = \lambda_2 = 1$ - algebraic multiplicity 2 ; $\lambda_3 = 2$

To find the eigenvectors solve $(A - 1.I)X = 0$ i.e.

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \underset{s\nearrow}{0} & 0 \end{array}\right) \qquad X = \begin{pmatrix} s \\ s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Eigenvectors are only defined up to a scale so we can choose $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Notice that we do not have a second eigenvector associated to $\lambda = 1$. The geometric multiplicity is 1! (This will lead to complications later).

For $\lambda_3 = 2$ we have:

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad X = s\begin{pmatrix} 1/4 \\ 1/2 \\ 1 \end{pmatrix} \rightarrow X_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

rem: The eigenvector could be normalized to length 1:

$\tilde{X}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\tilde{X}_3 = \frac{1}{\sqrt{21}}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$. Still a sign ambiguity remains.

Diff. Eq - § - Eigenvalues and Eigenvectors

Ex: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{pmatrix} = -\lambda^2(\lambda+2)$. $\quad \lambda_1=\lambda_2=0 \quad \lambda_3=-2$

$\lambda=0 \quad \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad X = \begin{pmatrix} s \\ t \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ Geometric multiplicity = 2 = Algebraic Multiplicity

$\lambda=-2 \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad X = \begin{pmatrix} -u \\ 3u \\ u \end{pmatrix} = u\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \quad X_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

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rem: For $n\times n$ matrix A, the characteric equation $\det(A-\lambda I)=0$ is a polynomial equation of degree n. By the FTA it has precisely n roots (counting with multiplicity).

An eigenvalue with algebraic multiplicity m has geometric multiplicity q s.t. $1 \le q \le m$. Thus simple eigenvalues $m=1$ also have geometric multiplicity $q=1$.

The eigenvectors corresponding to different eigenvalues $\lambda_1 \ne \lambda_2$ are linearly independent.

Ex: Find the eigenvalues and the eigenvectors of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2+1 \quad \lambda_1 = i \quad \lambda_2=-i$

$\lambda=i \quad \left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right) \quad X = \begin{pmatrix} is \\ s \end{pmatrix} \quad X_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda=-i \quad \left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right) \quad X = \begin{pmatrix} -it \\ t \end{pmatrix} \quad X_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Ex: Find the eigenvalues and the eigenvectors of $A = \begin{pmatrix} 3 & -2i \\ 2i & 0 \end{pmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & -2i \\ 2i & -\lambda \end{pmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1) \quad \lambda_1=4, \; \lambda_2=-1$

Diff. Eq. - § - Eigenvalues and Eigenvectors.

$\lambda = 4$ $\left(\begin{array}{cc|c} -1 & -2i & 0 \\ 2i & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2i & 0 \\ 0 & 0 & 0 \end{array}\right)$ $X = \begin{pmatrix} -2is \\ s \end{pmatrix}$ $X_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$

$\lambda = -1$ $\left(\begin{array}{cc|c} 4 & -2i & 0 \\ 2i & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -i/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $X = \begin{pmatrix} i/2\, t \\ t \end{pmatrix}$ $X_2 = \begin{pmatrix} i \\ 2 \end{pmatrix}$

rem: The matrix in the last example is complex but the eigenvalues are real.

Def: A matrix is self-adjoint (Hermitian) if $A^* = A$, where $A^* = \bar{A}^T$.

Ex: Matrices with real elements which are symmetric ($A^T = A$) are self-adjoint

Ex: $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2-i \\ 2+i & 7 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 1+i \\ 1-i & -i \end{pmatrix}$ $D = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 2 & 3 \end{pmatrix}$

Th: The eigenvalues and the eigenvectors of a Hermitian matrix have the following properties:

1. All eigenvalues are real.
2. The algebraic and geometric multiplicities of all eigenvalues match.
3. The eigenvectors corresponding to different eigenvalues are orthogonal.
4. Corresponding to an eigenvalue of multiplicity m it is always possible to select a set of m orthogonal associated eigenvectors.