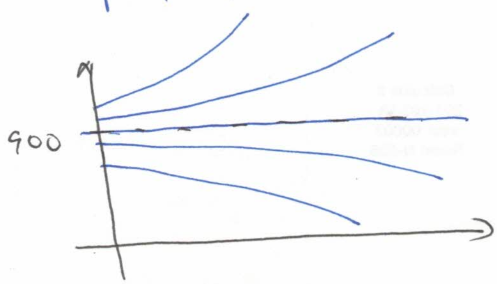
Diff Eq. - §2 - Solutions of DE's : Types of DE's

Q: What does it mean to solve analytically a DE?

Ex: Field mice and hawks: $\frac{dp}{dt} = 0.5p - 450 = \frac{p-900}{2}$

$\int \frac{dp}{p-900} = \int \frac{dt}{2}$, $\ln|p-900| = \frac{t}{2} + c$

$|p-900| = e^c e^{t/2}$; $p - 900 = \pm e^c e^{t/2}$; $p = 900 + ce^{t/2}$



Usually an initial point is specified, say $p(0) = 1050$ ← initial condition.
The DE together with the initial condition is called initial value problem.

Ex: Cont'd: $p(0) = 1050 = 900 + ce^{0/2} = 900 + c$ $c = 150$

$$\boxed{p(t) = 900 + 150e^{t/2}}$$

Ex: Consider the family of initial value problems

$$y' = ay - b, \quad y(0) = y_0.$$

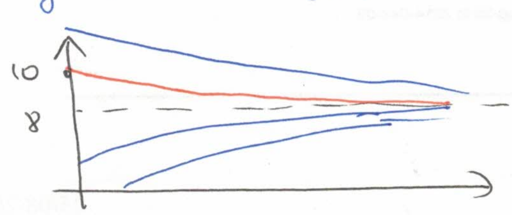
Sol: $\frac{dy}{dt} = a(y - b/a)$, $\int \frac{dy}{y - b/a} = \int a\,dt$, $\ln|y - b/a| = at + C$

$y - b/a = \pm e^C e^{at}$; $y = b/a + ce^{at}$ ← general solution

$y_0 = y(0) = b/a + c$, $c = y_0 - b/a$, $\boxed{y(t) = b/a + (y_0 - b/a)e^{at}}$

Ex: $y' = -\frac{1}{2}y + 4$, $a = -1/2$, $b = -4$: $y = -4/(-1/2) + ce^{-1/2 t} = 8 + ce^{-1/2 t}$

$y(0) = 10$ $y(t) = 8 + 2e^{-1/2 t}$



Out of all the integral curves we picked out the one which goes through the initial point.

Diff. Eq. - §2 - Solutions of DE; Types of DE's. (2)

(1) Ordinary diff. eq's: The unknown function depends on a single var.

e.g. $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 4y = 3e^t$, $y = y(t)$.

(1') Partial DE's: The unknown function depends on more than one variable, e.g. $f(x,y)$

$$2\frac{\partial^2 f}{\partial x^2} - 3\frac{\partial^2 f}{\partial x\partial y} + 4\frac{\partial^2 f}{\partial y^2} = \sin(x+y).$$

$$k^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(wave equation.)}$$

(2) Systems of DE's: (more than one unknown function).

$$\frac{dm}{dt} = +0.5m - 0.2h$$
$$\frac{dh}{dt} = 0.1h + 0.8m$$

Lotka-Volterra (predator-prey) type system.

(3) The order of a DE is the order of the highest derivative in the equation.

$$t y^{(5)}(t) + 3t^2 y'''(t) = -e^{-2t} \quad \leftarrow \text{order } 5.$$

$$3\frac{dy}{dt} = \left(\frac{dy}{dt}\right)^2 - 7t \quad \leftarrow \text{first order}$$; $y(t)^2 - 3y(t) + 4t^2 = -3$ (not a DE).

(4) An ordinary DE is linear if the equation only involves the derivatives $y^{(n)}, y^{(n-1)}, \dots, y', y$ in linear fashion.

$$3y'' - 4y' + 7y = 2e^t \quad \leftarrow \text{linear}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \quad \theta = \theta(t) \quad \leftarrow \text{nonlinear. (pendulum).}$$

$$y'' + 2yy' + y = 4t^2 \quad \leftarrow \text{nonlinear.}$$

(A) Existence of solutions: analytical; numerical approximations.
(B) Uniqueness of solutions: initial conditions.

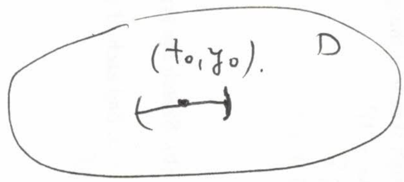
Th: (Cauchy-Peano existence theorem).

Let $f: D \to \mathbb{R}$, where D is open in $\mathbb{R}\times\mathbb{R}$ and f is continuous consider the initial value problem

$$y'(t) = f(t, y) \quad , y(t_0) = y_0 \quad , \quad (t_0, y_0) \in D.$$

It has a solution (local) $z: I \to \mathbb{R}$ where I is an nhood of t_0 in $\mathbb{R}$ st.

$$z'(t) = f(t, z(t)) \quad \forall t \in I.$$



Th: (Picard - Lindelöf) Suppose that f is uniformly Lipschitz continuous in y. Then the solution is unique on some interval containing t_0.