Diff. Eq. - §20 - Basic Theory of systems of 1^{st} order DE's

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \dots + p_{1n}(t)x_n + g_1(t) \\ &\dots \\ x_n' &= p_{n1}(t)x_1 + \dots + p_{nn}(t)x_n + g_n(t) \end{aligned} \quad \to \quad \vec{x}' = P(t)\vec{x} + g(t)$$

Consider first the homogeneous equation $\vec{x}' = P(t)\vec{x}$. Its solutions will be written as:

$$x^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \dots, x^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}.$$

Th: (Principle of superposition).
If the vector functions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions of $\vec{x}' = P(t)\vec{x}$, then any linear combination $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is also a solution.

Ex: $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x}$ i.e. $\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$

Then $\vec{x}^{(1)}(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ and $\vec{x}^{(2)}(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ are solutions. Then so is

$$\vec{x} = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t).$$

Cor: If $\vec{x}^{(1)}, \dots, \vec{x}^{(k)}$ are solutions of $\vec{x}' = P(t)\vec{x}$ so is $\vec{x} = c_1\vec{x}_1 + \dots + c_k x^{(k)}$.

rem: As in the case of n'th order DE we expect n solutions $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$.
Arrange them in a matrix as columns:

$$\vec{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}.$$

The solutions $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are linearly independent at a point iff the Wronskian $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}] = \det \vec{X}(t)$ is nonzero there.

Th: If the vector functions $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are linearly independent solutions of $\vec{x}' = P(t)\vec{x}$ for each point of the interval $\alpha < t < \beta$, then each solution of this system can be expessed uniquely as a linear combination

$$\vec{x} = c_1\vec{x}^{(1)}(t) + \dots + c_n\vec{x}^{(n)}(t).$$

Diff. Eq - § - Basic Theory of systems of 1st order DE

In This case $\vec{x} = c_1\vec{x}_1 + \dots + c_n\vec{x}_n$ is called The general solution; $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ form a fundamental set of solutions.

Prop: If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are solutions of $\vec{x}' = P(t)\vec{x}$ on an interval $\alpha < t < \beta$, Then on This interval $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}]$ is either identically zero or else never vanishes.

Pr: Abel's Formula: $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}] = W(t) = c\, e^{\{\int [p_{11}(t) + \dots + p_{nn}(t)]\, dt\}}$.

Cor: If $\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)$ are linearly independent at one point of The interval on which They are defined Then They are linearly independent on The whole interval.

rem: A fundamental set of solutions for $\vec{x}'(t) = P(t)\vec{x}(t)$ could be found using The Existence Theorem with initial conditions

$$x(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ Then } x(t_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \text{and finally } x(t_0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$