Diff Eq - §21 - Homogeneous Linear Systems with constant coefficients

We will consider systems of the form $\vec{x}' = A\vec{x}$, where A is constant $n \times n$ matrix.

rem: In the scalar case $x' = ax$, the solution is $x(t) = ce^{at}$.

rem: Now in the general case we will search for a solution of the form $\vec{x} = \vec{\xi} e^{rt}$. Then $r\vec{\xi}e^{rt} = A\vec{\xi}e^{rt} \Rightarrow A\vec{\xi} = r\vec{\xi} \Rightarrow r$ is an eigenvalue and $\vec{\xi}$ is the corresponding eigenvector.

Ex: $x_1' = x_1 + x_2$ i.e. $\vec{x}' = A\vec{x}$, $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.
$x_2' = 4x_1 + x_2$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1).$$

$$\lambda_1 = 3 \quad \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x = \begin{pmatrix} 1/2\, s \\ s \end{pmatrix} \to x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x = \begin{pmatrix} -1/2\, t \\ t \end{pmatrix} \Rightarrow x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The solution is: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

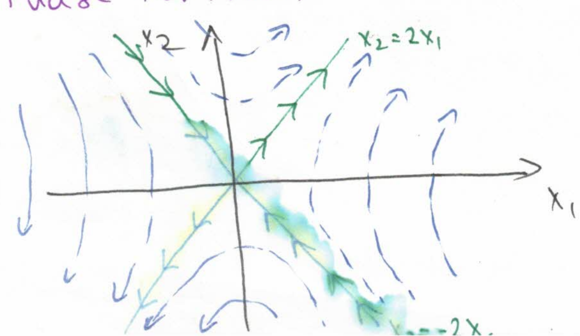
rem: Here is (again) why this is a solution.

$$\vec{x}'(t) = c_1 e^{3t}\, 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}(-1)\begin{pmatrix} 1 \\ -2 \end{pmatrix} = c_1 e^{3t} A\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = A\vec{x}(t).$$

Ex cont'd: The Wronskian of these solutions: $\vec{x}^{(1)}(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{x}^{(2)}(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is

$$W[\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)] = \det\begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} = -4e^{2t} \neq 0,$$ so indeed they form a fundamental set of solutions.

Phase Portrait (Directional Field with few trajectories).



- saddle point
- all solutions are eventually asymptotic to the line $x_2 = 2x_1$ as $t \to \infty$.
- note this is not a graph as a function of t.

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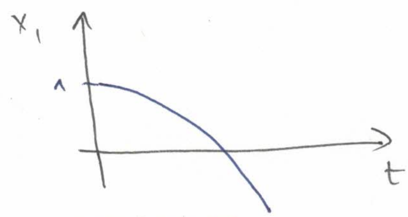
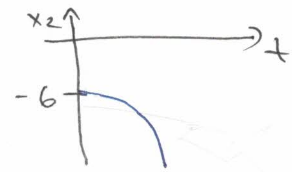
Ex cont'd: solve $\vec{x}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}(t)$ with initial conditions $\vec{x}(0) = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \Rightarrow c_1 = -1,\ c_2 = 2$

$\vec{x}(t) = -e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$x_1(t) = -e^{3t} + 2e^{-t}, \quad x_2(t) = -2e^{3t} - 4e^{-t}$



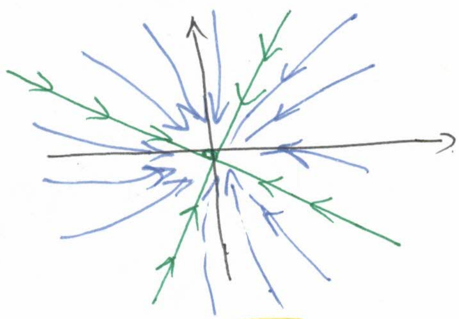


Ex: Consider the system $\vec{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \vec{x}$

$\det(A - \lambda I) = \det \begin{pmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{pmatrix} = \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4)$

$\lambda_1 = -1 \quad \left(\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x = \begin{pmatrix} s/\sqrt{2} \\ s \end{pmatrix} \quad x_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

$\lambda_2 = -4 \quad \left(\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x = \begin{pmatrix} -\sqrt{2}t \\ t \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ -1/\sqrt{2} \end{pmatrix}$



$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1/\sqrt{2} \end{pmatrix}$

– now the origin is a node (attractor).

Constr: $\vec{x}' = A\vec{x}$, A - $n\times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Assume A is real valued. Then the possibilities are:

1. All eigenvalues are real and different from each other.
2. Some eigenvalues occur in complex conjugate pairs.
3. Some eigenvalues, either real or complex, are repeated.

In the case 1 we have n real eigenvectors $\vec{\xi}_1, \dots, \vec{\xi}_n$ which are

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linearly independent. We have solutions

$$\vec{x}^{(1)}(t) = e^{\lambda_1 t}\vec{\xi}_1, \dots, \vec{x}^{(n)} = e^{\lambda_n t}\vec{\xi}_n$$

Their Wronskian is

$$W[\vec{x}^{(1)}, \dots, x^{(n)}](t) = \det\left[e^{\lambda_1 t}\vec{\xi}_1, \dots, e^{\lambda_n t}\vec{\xi}_n\right] =$$

$$= e^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)t} \det\left[\vec{\xi}_1, \dots, \vec{\xi}_n\right] \neq 0,$$

since $\vec{\xi}_1, \dots, \vec{\xi}_n$ are linearly independent. The general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{\xi}_1 + \dots + c_n e^{\lambda_n t}\vec{\xi}_n. \quad (*)$$

rem: If A is real and symmetric (Hermitian), the eigenvalues are real, there are n eigenvectors $\vec{\xi}_1, \dots, \vec{\xi}_n$ and they are orthogonal. In this case $(*)$ again is the general solution (multiplicities will match).

ex: Find the general solution of $\vec{x}' = A\vec{x}$, $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

sol: $\lambda_1 = 2$, $\vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\lambda_2 = \lambda_3 = -1$, $\vec{\xi}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{\xi}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

rem: Cases (2), (3) will be treated in the next two sections.