Diff Eq - #22 - Complex eigenvalues

Consider the linear homogeneous system of DE's $x' = Ax$.
Even when the matrix A is real, its eigenvalues/eigenvectors could be complex.

Ex: $\vec{x}\,' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \vec{x}$

The eigenvalue-eigenvector pairs are

$$\lambda_1 = -\frac{1}{2} + i, \quad \underline{X}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}; \qquad \lambda_2 = -\frac{1}{2} - i, \quad \underline{X}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Fundamental set of solutions is

$$\vec{x}^{(1)}(t) = e^{(-1/2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}; \qquad \vec{x}^{(2)}(t) = e^{(-1/2-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Notice that

$$\vec{x}^{(1)}(t) = e^{(-1/2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-t/2}(\cos t + i \sin t)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\Rightarrow \vec{x}^{(1)}(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

similarly,

$$\vec{x}^{(2)}(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - i e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

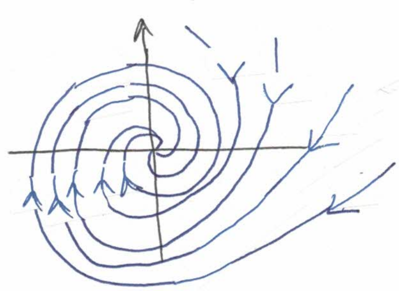
We can form a new fundamental set of solutions consisting of the real and imaginary parts of $\vec{x}^{(1)}(t)$ (and $\vec{x}^{(2)}(t)$):

$$\vec{u}(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \vec{v}(t) = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Let's check the Wronskian:

$$W(\vec{u}, \vec{v}) = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} = e^{-t} \neq 0.$$

The phase portrait or the system is:



Spiral point - attractor.

Note that $\begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

so the rotation is clockwise.

Diff Eq – H22 – Complex eigenvalues

In general, for a 2d system $\vec{x}' = A\vec{x}$, where the eigenvalues are complex: $\lambda_1 = \lambda + i\mu$, $\lambda_2 = \bar{\lambda}_1 = \lambda - i\mu$, the eigenvectors will also be complex conjugate of each other:

$$X_1 = Y + iZ, \quad X_2 = \bar{X}_1 = Y - iZ, \quad Y, Z \in \mathbb{R}^2$$

We can always from a real fundamental set:

$$\vec{x}^{(1)}(t) = e^{\lambda_1 t} X_1 = e^{(\lambda + i\mu)t}(Y + iZ) =$$

$$= e^{\lambda t}(\cos\mu t + i\sin\mu t)(Y + iZ) =$$

$$= e^{\lambda t}(Y\cos\mu t - Z\sin\mu t) + i e^{\lambda t}(Y\sin\mu t + Z\cos\mu t)$$

$$\left.\begin{aligned} \vec{u}(t) &= e^{\lambda t}(Y\cos\mu t - Z\sin\mu t) \\ \vec{v}(t) &= e^{\lambda t}(Y\sin\mu t + Z\cos\mu t) \end{aligned}\right\} \text{Fundamental set.}$$

Even more generally for an n-dimensional system $\vec{x}' = AX$, with real coefficient matrix $A \in \mathbb{R}^{n\times n}$, the compex eigenvalues come in complex conjugate pairs and we can always replace the pairs of complex conjugate solutions with their real and imaginary parts obtaining a real basis.

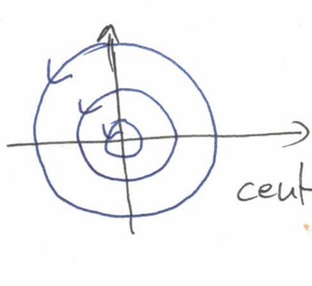
Ex: Consider the system

$$x' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$$

The eigenvalue-eigenvector pairs are: $\lambda_1 = i$, $X_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$; $\lambda_2 = -i$, $X_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\vec{x}^{(1)}(t) = e^{it}\begin{pmatrix} 1 \\ -i \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\vec{u}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad \vec{v}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$



center.

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ so the rotation is counterclockwise.

DIFERENTIAL EQUATIONS, BIFURCATIONS

Consider the following system of DE's depending on a parameter $\alpha \in \mathbb{R}$.

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$$

Describe how the phase portrait depend on α.

Solution: The characteristic polynomial of the coefficient matrix is

$$\det \begin{pmatrix} \alpha - \lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 - \alpha\lambda + 1 = 0$$

The two eigenvalues are

$$\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

There are the following possibilities depending on the value of the discriminant.

(1) $\alpha^2 - 4 > 0$. Then both eigenvalues are real and distinct.
   - (a) $\alpha < -2$. Then $\lambda_1 < 0, \lambda_2 < 0 \to$ stable node.
   - (b) $\alpha > 2$. Then $\lambda_1 > 0, \lambda_2 > 0 \to$ unstable node.

(2) $\alpha^2 - 4 < 0$. Then we have two complex conjugate eigenvalues.
   - (a) $-2 < \alpha < 0 \to$ stable spiral point.
   - (b) $0 < \alpha < 2 \to$ unstable spiral point.
   - (c) $\alpha = 0 \to$ center.

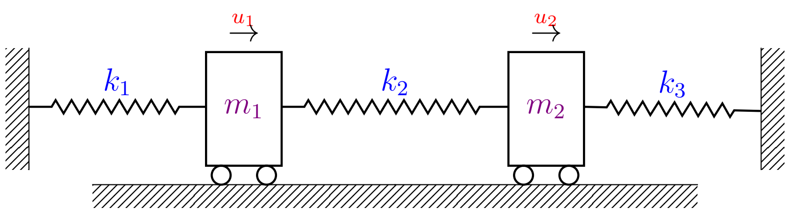
(3) $\alpha = -2 \to$ stable improper node.

(4) $\alpha = 2 \to$ unstable improper node.

Try to imagine how the phase portrait types morph into each other as α changes from -3 to 3. Can you see how the improper nodes separate nodes from spirals; or how the center separates the unstable spirals from the stable spirals?

DIFERENTIAL EQUATIONS, MASSES AND SPRINGS

Consider the two-mass, three-spring system drawn below, with no external forces.



Let $m_1 = 2, m_2 = 9/4, k_1 = 1, k_2 = 3$, and $k_3 = 15/4$.
a) Convert the dynamical equations of this system to four first order DE's and then write them in the form $u' = Au$.
b) Using software, find the eigenvalues and the eigenvectors of A.
c) Write down the general solution of the system.
d) Describe the four fundamental modes of vibration as four-vectors of functions and also in English.
e) For each fundamental mode draw graphs of the displacements u_1 and u_2 versus t on the same graph.

Solution: The equations of motion read

$$m_1 u_1'' = -(k_1 + k_2)u_1 + k_2 u_2$$
$$m_2 u_2'' = k_2 u_1 - (k_2 + k_3)u_2$$

Let's transform them into a system of four equations of 1st order. The new variables are defined as follows:

$$y_1 = u_1, \quad y_2 = u_2, \quad y_3 = u_1', \quad y_4 = u_2'$$

In terms of these variables the equations of motion read

$$y_1' = y_3, \quad y_2' = y_4$$
$$m_1 y_3' = -(k_1 + k_2)y_1 + k_2 y_2$$
$$m_2 y_4' = k_2 y_1 - (k_2 + k_3)y_2$$

Plugging in the specific values we have

$$\begin{aligned} y_1' &= y_3, \quad y_2' = y_4 \\ 2y_3' &= -4y_1 + 3y_2 \\ \frac{9}{4}y_4' &= 3y_1 - \frac{27}{4}y_2 \end{aligned}$$

In matricial form the system is

$$\mathbf{y}' = A\mathbf{y}, \qquad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix}$$

Now we have to compute the eigenvalue - eigenvector pairs (using software). The characteristic polynomial is:

$$\det(A - \lambda I) = \lambda^4 + 5\lambda^2 + 4 = (\lambda^2 + 1)(\lambda^2 + 4)$$

The eigenvectors-eigenvalues come in complex conjugate pairs:

$$\lambda_1 = i, \quad \xi_1 = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}; \qquad \lambda_2 = -i, \quad \xi_2 = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}$$

$$\lambda_3 = 2i, \quad \xi_3 = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}; \qquad \lambda_4 = -2i, \quad \xi_4 = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}$$

Next we rewrite the basic solutions in real terms

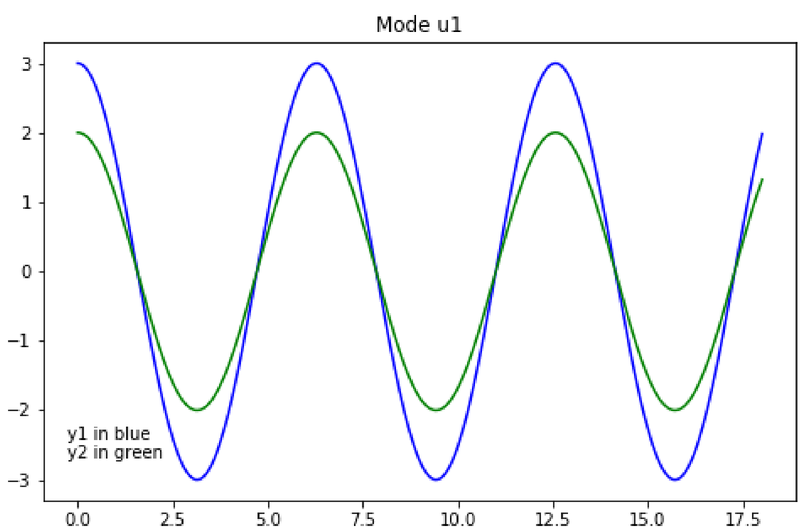
$$\begin{aligned} \mathbf{x}^{(1)}(t) = e^{it} \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} &= (\cos t + i \sin t) \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} \\ &= \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + i \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} = \mathbf{u}^{(1)}(t) + i\mathbf{v}^{(1)}(t) \end{aligned}$$

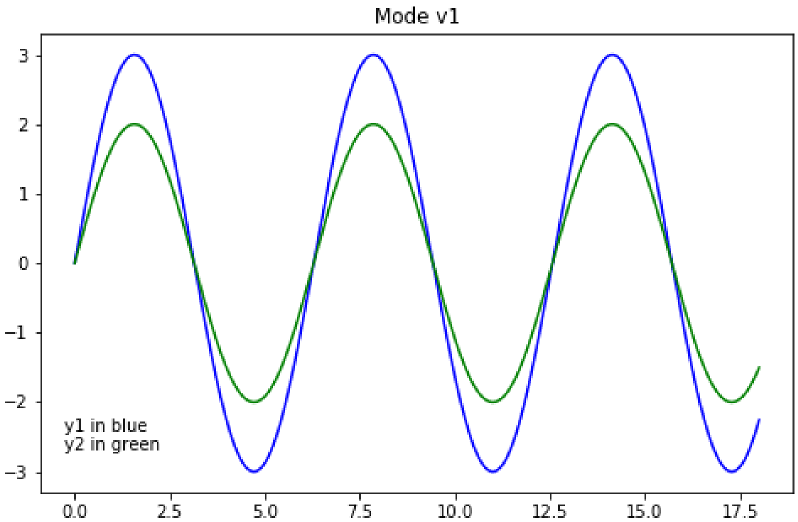
$$\mathbf{x}^{(3)}(t) = e^{2it}\begin{pmatrix} 3 \\ -4 \\ 6i \\ 8i \end{pmatrix} = (\cos 2t + i \sin 2t)\begin{pmatrix} 3 \\ -4 \\ 6i \\ 8i \end{pmatrix}$$
$$= \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + i\begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} = \mathbf{u}^{(2)}(t) + i\mathbf{v}^{(2)}(t)$$

The general solution is

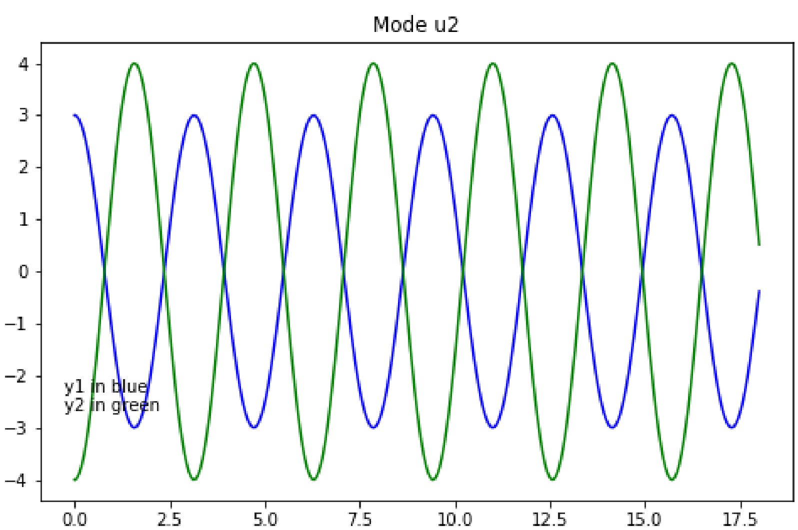
$$\mathbf{y}(t) = c_1\mathbf{u}^{(1)} + c_2\mathbf{v}^{(1)} + c_3\mathbf{u}^{(2)} + c_4\mathbf{v}^{(2)}$$

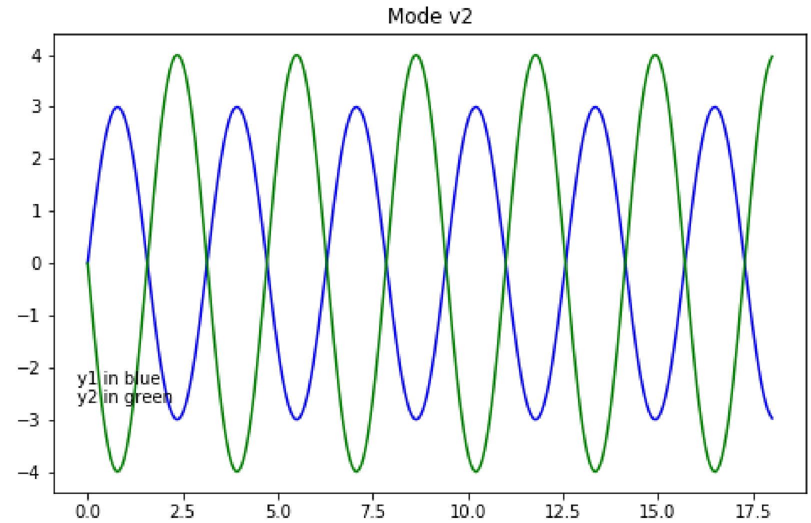
The fundamental modes are $\mathbf{u}^{(1)}, \mathbf{v}^{(1)}, \mathbf{u}^{(2)}, \mathbf{v}^{(2)}$. Notice that for $\mathbf{u}^{(1)}, \mathbf{v}^{(1)}$ we have $y_1 = 3/2\ y_2$ and at the same time $y_3 = 3/2\ y_4$. This means that in these two modes the two masses are moving together (synchronously) in the same direction, but the first mass is moving 3/2 times as far as the first mass. The frequency of these two modes is 1 (period 2π). For the mode $\mathbf{u}^{(1)}$ the phase difference between the positions and the velocities is $-\pi/2$ and for the mode $\mathbf{v}^{(1)}$ this phase difference is $+\pi/2$.





For $\mathbf{u}^{(2)}, \mathbf{v}^{(2)}$ we have $y_1 = -3/4\ y_2$ and at the same time $y_3 = -3/4\ y_4$. This means that in these two modes the two masses are moving in opposite directions (synchronously) and the first mass is moving 3/4 times as far as the first mass. The frequency of these two modes is 2 (period π). For the mode $\mathbf{u}^{(2)}$ the phase difference between the positions and the velocities is $-\pi/2$ and for the mode $\mathbf{v}^{(2)}$ this phase difference is $+\pi/2$.





For a general initial conditions we will have a linear combination of all four fundamental nodes. Here is what the dynamics of the displacements of the two masses looks like under the linear combination $2u^{(1)}(t) - v^{(1)}(t) - 3u^{(2)}(t) + v^{(2)}(t)$.