Diff Eq - H23 - Fundamental matrices and matrix exponents

Consider the linear homogeneous system of 1^{st}-order DE's:

$$\vec{x}'(t) = P(t)\, x(t)$$

and let $\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)$ be a fundamental set of solutions. The matrix

$$\Psi(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}$$

is said to be a fundamental matrix for this system of DE's. Since the solutions in the fundamental set are linearly independent the fundamental matrix $\Psi(t)$ is non-singular.

ex: The system

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

has a fundamental set of solutions

$$\vec{x}^{(1)} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \quad \vec{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

A fundamental matrix is

$$\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

If $\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)$ is a fundamental set of solutions, the general solution is of the form $\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_n \vec{x}^{(n)}(t)$. In matricial form

$$\vec{x}(t) = \Psi(t)\, \vec{c}, \quad \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

If initial conditions $\vec{x}(t_0) = \vec{x}^0$ are given we have

$$\Psi(t_0)\vec{c} = \vec{x}^0 \Rightarrow \vec{c} = \Psi(t_0)\, \vec{x}^0$$

$$\boxed{\vec{x}(t) = \Psi(t)\,\Psi^{-1}(t_0)\, \vec{x}^0}$$

A special fundamental matrix $\Phi(t)$ is obtained if we solve (n times) with initial conditions $\Phi(t_0) = I_n$.

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$$\vec{x}(t) = \Phi(t)\,\Phi^{-1}(t_0)\,\vec{x}^{\,0} = \Phi(t)\,\vec{x}^{\,0}$$

Thus $\Phi(t)$ represents the transformation of the initial state $\vec{x}^{\,0}$ to the solution $\vec{x}(t)$ for arbitrary time t.

ex: The system $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ has a general solution

$$\vec{x}(t) = c_1 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

To find the special fundamental matrix we solve with two different initial conditions

$$\vec{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2 = \frac{1}{2}$$

$$\vec{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow c_1 = \frac{1}{4},\ c_2 = -\frac{1}{4}$$

Thus

$$\Phi(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

Say initial condition

$$\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is given. Then

$$\vec{x}(t) = \Phi(t)\,\vec{x}(0) = \begin{pmatrix} \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{3t} - \frac{3}{2}e^{-t} \end{pmatrix}$$

Matrix exponents

Consider $n \times n$ matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$ and with precisely n corresponding eigenvectors $\xi^{(1)}, \dots, \xi^{(n)}$ (so no eigenvectors are 'missing'). Let

$$T = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}$$

We have:

$$AT = \begin{pmatrix} \lambda_1 \xi_1^{(1)} & \cdots & \lambda_n \xi_1^{(n)} \\ \vdots & & \vdots \\ \lambda_1 \xi_n^{(1)} & \cdots & \lambda_n \xi_n^{(n)} \end{pmatrix} = T \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

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With the diagonal matrix $D = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ in the case of no missing 'eigenvectors' we have

$$AT = TD$$

and since T is invertible

$$T^{-1}AT = D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Thus A is similar to a diagonal matrix. We say that the matrix A is diagonalizable.

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ex: $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$; $\lambda_1 = 3$, $\xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = -1$, $\xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$T = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$, $T^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$

$$T^{-1}AT = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Consider the linear system $\vec{x}' = A\vec{x}$ with A diagonalizable.
Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues and let $\xi^{(1)}, \dots, \xi^{(n)}$ be the corresponding eigenvectors. Define new variables

$$\vec{y} = T^{-1}\vec{x}, \quad \text{i.e. } \vec{x} = T\vec{y}, \quad T = (\xi^{(1)}, \dots, \xi^{(n)})$$

Then $\vec{x}' = (T\vec{y})' = T\vec{y}' = AT\vec{y}$ since $\vec{x}' = A\vec{x}$.

$$\vec{y}' = T^{-1}AT\vec{y}, \quad \vec{y}' = D\vec{y}$$

The system has split into variables which are no longer coupled

$$y_1' = \lambda_1 y_1, \quad \dots, \quad y_n' = \lambda_n y_n.$$

But all these DE's we can easily solve

$$y_1(t) = e^{\lambda_1 t} y_1(0), \quad \dots, \quad y_n(t) = e^{\lambda_n t} y_n(0)$$

Let

$$Q(t) = \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix}$$

We have $\vec{y}(t) = Q(t)\,\vec{y}(0)$ and hence

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$$T^{-1}\vec{x}(t) = Q(t)T^{-1}\vec{x}(0) \Rightarrow \vec{x}(t) = TQ(t)T^{-1}\vec{x}(0)$$

Cor: The matrix exponential for the system $\vec{x}' = A\vec{x}$ is

$$\boxed{e^{At} = \Phi(t) = TQ(t)T^{-1}} \quad . \quad Q(t) = \operatorname{diag}\{e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}.$$

Ex. $\vec{x}' = A\vec{x}$, $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

$\lambda = 3$ $\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = -1$, $\xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$e^{At} = TQ(t)T^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

Matrix exponentials give us a very well organized and efficient way of solving linear, homogeneous systems of DE's with constant coefficients.

DIFERENTIAL EQUATIONS, MATRIX EXPONENT

Consider the system of linear DE's

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

i) Compute the fundamental matrix $\Phi(t) = e^{At}$.
ii) Solve this system with the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Solution: Start by computing the eigenvalue - eigenvector pairs:

$$\det \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = (\lambda - 4)(\lambda + 1)$$

$$\lambda_1 = 4, \xi^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \lambda_2 = -1, \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The matirx of eigenvectors is

$$T = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$$

The matrix exponential is

$$\begin{aligned} e^{At} = TQ(t)T^{-1} &= \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \\ &= \frac{1}{5} \begin{pmatrix} 3e^{4t} + 2e^{-t} & 3e^{4t} - 3e^{-t} \\ 2e^{4t} - 2e^{-t} & 2e^{4t} + 3e^{-t} \end{pmatrix} \end{aligned}$$

The solution satisfying the given initial conditions is

$$\begin{aligned} \mathbf{x}(t) = e^{At}\mathbf{x}(0) &= \frac{1}{5} \begin{pmatrix} 3e^{4t} + 2e^{-t} & 3e^{4t} - 3e^{-t} \\ 2e^{4t} - 2e^{-t} & 2e^{4t} + 3e^{-t} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 3e^{4t} - 8e^{-t} \\ 2e^{4t} + 8e^{-t} \end{pmatrix} \end{aligned}$$

DIFERENTIAL EQUATIONS, FUNCTION OF A MATRIX

For the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

compute $\sin\left(\frac{\pi}{4}A\right)$.

Solution: For any diagonalizable matrix A

$$A = TDT^{-1}, \quad D = diag\{\lambda_1, \ldots, \lambda_n\}$$

and any analytical function $f(x)$ we have by definition

$$f(A) = Tf(D)T^{-1} = T\ diag\{f(\lambda_1), \ldots, f(\lambda_n)\}\ T^{-1}$$

The function is implemented on the eigenvalues.

The eigenvalue - eigenvector pairs are:

$$\lambda_1 = 4, \xi^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \lambda_2 = -1, \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The matirx of eigenvectors is

$$T = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$$

We have

$$\begin{aligned} \sin\left(\frac{\pi}{4}A\right) &= Tf(D)T^{-1} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \sin\left(\frac{4\pi}{4}\right) & 0 \\ 0 & \sin\left(\frac{-\pi}{4}\right) \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \\ &= Tf(D)T^{-1} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \\ &= \frac{1}{5\sqrt{2}} \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \end{aligned}$$