Diff Eq - H24 - Repeated Eigenvalues
Jordan normal form.

Ex: Consider the system

$$\vec{x}' = A\vec{x}, \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 \qquad \lambda_1 = \lambda_2 = 2$$

$$\lambda_{1,2} = 2 \quad \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad \xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Algebraic multiplicity $\to 2$; Geometric multiplicity $\to 1$ $\Rightarrow$ missing eigenvector.

We have only one solution $\vec{x}^{(1)}(t) = e^{2t}\xi_1$.

Ansatz: We will search for a second solution of the form

$$\vec{x}^{(2)}(t) = te^{2t}\xi_1 + e^{2t}\eta$$

Let's plug the ansatz in the system of DE's we have

$$\vec{x}^{(2)\prime} = e^{2t}\xi_1 + \cancel{2te^{2t}\xi_1} + 2e^{2t}\eta = A\left(\cancel{te^{2t}\xi_1} + e^{2t}\eta\right)$$

$$\Rightarrow \cancel{e^{2t}}\xi_1 + 2\cancel{e^{2t}}\eta = A\cancel{e^{2t}}\eta \Rightarrow (A-2I)\eta = \xi_1$$

The generalized eigenvector η is a solution of a linear system

$$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cancel{+ s\xi_1}$$

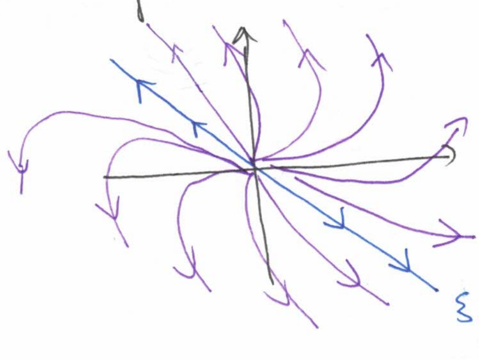
$\uparrow s$

The general solution is

$$\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$$

$$\vec{x}(t) = c_1e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]$$

The phase portrait is an unstable improper node:



Diff Eq - H24 - Repeated eigenvalues

Jordan normal form

We solve twice for c_1 and c_2 with the initial conditions $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ to find the matrix exponential $\Phi(t) = e^{At}$.

$$\begin{pmatrix}1\\0\end{pmatrix} = c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}-1\\0\end{pmatrix} \Rightarrow c_1 = 0 \quad c_2 = -1$$

$$\tilde{x}^{(1)}(t) = -te^{2t}\begin{pmatrix}1\\-1\end{pmatrix} - e^{2t}\begin{pmatrix}-1\\0\end{pmatrix} = e^{2t}\begin{pmatrix}1-t\\t\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}-1\\0\end{pmatrix} \Rightarrow c_1 = -1 \quad c_2 = -1$$

$$\tilde{x}^{(2)}(t) = -e^{2t}\begin{pmatrix}1\\-1\end{pmatrix} - \left[te^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + e^{2t}\begin{pmatrix}-1\\0\end{pmatrix}\right] = e^{2t}\begin{pmatrix}-t\\1+t\end{pmatrix}$$

$$e^{At} = e^{2t}\begin{pmatrix}1-t & -t\\ t & 1+t\end{pmatrix}$$

We can also attempt to diagonalize the coefficient matrix A using the eigenvector ξ_1 and the generalized eigenvector η

$$T = (\xi_1 \ \eta) = \begin{pmatrix}1 & -1\\-1 & 0\end{pmatrix}, \quad T^{-1} = \begin{pmatrix}0 & -1\\-1 & -1\end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix}0 & -1\\-1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 0\end{pmatrix} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix} = J$$

J is the Jordan normal form of the undiagonalizable matrix A.

What is e^J?

$$J^2 = \begin{pmatrix}4 & 4\\0 & 4\end{pmatrix} = \begin{pmatrix}2^2 & 2\cdot 2^1\\0 & 2^2\end{pmatrix}; \quad J^3 = \begin{pmatrix}2^3 & 3\cdot 2^2\\0 & 2^3\end{pmatrix}; \quad J^4 = \begin{pmatrix}24 & 4\cdot 2^3\\0 & 24\end{pmatrix}$$

$$J^n = \begin{pmatrix}2^n & n\cdot 2^{n-1}\\0 & 2^n\end{pmatrix}$$

$$e^{Jt} = \sum_{n=0}^{\infty}\frac{(Jt)^n}{n!} = \begin{pmatrix}\sum_{n=0}^{\infty}\frac{2^n}{n!}t^n & \sum_{n=0}^{\infty} n\frac{2^{n-1}}{n!}t^n\\ 0 & \sum_{n=0}^{\infty}\frac{2^n}{n!}t^n\end{pmatrix} =$$

$$= \begin{pmatrix}e^{2t} & te^{2t}\\0 & e^{2t}\end{pmatrix}$$

Diff Eq - H24 - Repeated eigenvalues. Jordan normal form.

$$e^{At} = \sum_{n=0}^{\infty} (TJT^{-1})^n/n! = T\left(\sum_{n=0}^{\infty} \frac{J^n}{n!}\right)T^{-1} = Te^{Jt}T^{-1}$$

$$e^{At} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = e^{2t}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} =$$

$$= e^{2t}\begin{pmatrix} 1 & t-1 \\ -1 & -t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = e^{2t}\begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \quad \text{(same as before).}$$

Here are the possible Jordan forms of 3×3 matrices:

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

Coming back to the 2×2 case $\vec{x}' = A\vec{x}$ with $\lambda_1 = \lambda_2 = \lambda$ and only one eigenvector ξ_1 the general solution is

$$\vec{x}(t) = c_1 e^{\lambda t}\xi_1 + c_2\left[te^{\lambda t}\xi_1 + e^{\lambda t}\eta\right],$$

where the generalized eigenvector η satisfies

$$(A - \lambda I)\eta = \xi_1$$

or equivalently

$$(A-\lambda I)^2\eta = (A-\lambda I)\xi_1 = 0.$$

DIFERENTIAL EQUATIONS, REPEATED EIGENVALUES

Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ 30 \end{pmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ -4 & 1-\lambda & 0 \\ 3 & 6 & 2-\lambda \end{pmatrix} = -(\lambda-1)^2(\lambda-2)$$

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 2$$

Next we find the eigenvectors

$$\lambda = 1, \left(\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 3 & 6 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/6 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad \xi_1 = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}$$

The second eigenvector is 'missing', so we will solve for a generalized eigenvector

$$(A - \lambda_1 I)\eta = \xi_1, \quad \left(\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & -1 \\ 3 & 6 & 1 & 6 \end{array} \right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 1/6 & 7/8 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad \eta = \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix} + s\xi_1$$

Ignoring the term in η proportional to ξ_1 we have

$$\eta = \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix}$$

For the third eigenvalue we have

$$\lambda = 2, \left(\begin{array}{rrr|r} -1 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 3 & 6 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\xi_1 + c_2[te^{\lambda_1 t}\xi_1 + e^{\lambda_1 t}\eta] + c_3 e^{\lambda_3 t}\xi_3$$

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} + c_2 \left[te^t \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} + e^t \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix} \right] + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

To satisfy the initial conditions we solve

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} + c_2 \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 30 \end{pmatrix}$$

$$c_1 = -11/2,\ c_2 = -4,\ c_3 = 3$$

Substituting the coefficients back in the general solution we get

$$\mathbf{x}(t) = -\frac{11}{2} e^t \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} - 4 \left[te^t \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} + e^t \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix} \right] + 3e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

After a simplification

$$\mathbf{x}(t) = e^t \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix} + te^t \begin{pmatrix} 0 \\ -4 \\ 24 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$