Diff Eq - §25 - Nonhomogeneous linear system

Consider a nonhomogeneous linear system of 1st order DE's:

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

The general solution can be expressed as

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_n x^{(n)}(t) + \vec{v}(t),$$

where

$$c_1 \vec{x}^{(1)}(t) + \dots + c_n x^{(n)}(t)$$

is the general solution of the homogeneous system $\vec{x}' = P(t)\vec{x}$ and $\vec{v}(t)$ is a particular solution of the nonhomogeneous system.
There are several methods for determining $\vec{v}(t)$.

Diagonalization.

Consider a system with constant coefficients

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

and let A be a diagonalizable matrix. Consider the matrix of eigenvectors

$$T = (\xi^{(1)}, \dots, \xi^{(n)})$$

Define new variables

$$\vec{y} = T^{-1}\vec{x}, \qquad \vec{x} = T\vec{y}$$

In the new variables the system becomes

$$T\vec{y}' = AT\vec{y} + \vec{g}(t) \Rightarrow \vec{y}' = T^{-1}AT\vec{y} + T^{-1}g(t)$$

$$\vec{y}' = \Lambda\vec{y} + \vec{h}(t), \quad \vec{h}(t) = T^{-1}g(t), \quad \Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$$

Since D is a diagonal matrix, the new system of DE's is a system of uncoupled equations

$$y_i'(t) = \lambda_i y_i(t) + h_i(t).$$

Using the method of integrating factors we can solve these equations

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$$y_i(t) = e^{\lambda_i t}\int e^{-\lambda_i s} h_i(s)\,ds + c_i e^{\lambda_i t}$$

The solution in terms of the original variables is obtained by changing the variables back:

$$\vec{x}(t) = T\vec{y}(t).$$

Ex: Consider

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = A\vec{x} + \vec{g}(t).$$

The eigenvalue-eigenvector pairs are

$$\lambda_1 = -3, \quad \xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \qquad \lambda_2 = -1, \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The change of variables matrix and its inverse are

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \vec{y} = T^{-1}\vec{x}$$

In the new variables the system reads

$$\vec{y}' = D\vec{y} + T^{-1}\vec{g}(t) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\vec{y} + \frac{1}{2}\begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}$$

In components the equations are

$$y_1' + 3y_1 = e^{-t} - \frac{3}{2}t, \quad y_2' + y_2 = e^{-t} + \frac{3}{2}t$$

and the solutions are

$$y_1 = e^{-3t}\int e^{3s}\left(e^{-s} - \frac{3}{2}s\right)ds + c_1 e^{-3t}$$

$$y_1 = \frac{e^{-t}}{2} - \frac{t}{2} + \frac{1}{6} + c_1 e^{-3t}$$

$$y_2 = e^{-t}\int e^{s}\left(e^{-s} + \frac{3}{2}s\right)ds + c_2 e^{-t}$$

$$y_2 = te^{-t} + \frac{3t}{2} - \frac{3}{2} + c_2 e^{-t}$$

Changing the variables back we have

$$\vec{x} = T\vec{y} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

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$$\vec{x}(t) = c_1 e^{-3t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} +$$

$$+ te^{-t}\begin{pmatrix}1\\1\end{pmatrix} + t\begin{pmatrix}1\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}$$

Method of undetermined coefficients.

Ex: $\vec{x}' = \begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\vec{x} + \begin{pmatrix}2e^{-t}\\3t\end{pmatrix}$ $\qquad \lambda_1 = -3,\ \lambda_2 = -1$

An ansatz for a particular solution will be: (notice $\lambda_2 = -1$)

$$\vec{y}(t) = te^{-t}\vec{a} + e^{-t}\vec{b} + t\vec{c} + \vec{d}$$

Plugging into the differential equation we obtain a system of equations for the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. The solution is

$$\vec{y}(t) = te^{-t}\begin{pmatrix}1\\1\end{pmatrix} + e^{-t}\begin{pmatrix}0\\-1\end{pmatrix} + t\begin{pmatrix}1\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}$$

Variation of parameters.

Consider the nonhomogeneous system

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

Assume that a fundamental matrix $\Psi(t)$ for the homogeneous system has been found. The general solution for the homogeneous system is $\vec{x}(t) = \Psi(t)\vec{c}$.

An ansatz for the nonhomogeneous system is

$$\vec{x}(t) = \Psi(t)\vec{u}(t)$$

Plugging into the equation we have

$$\cancel{\Psi'(t)\vec{u}(t)} + \Psi(t)\vec{u}'(t) = \cancel{P(t)\Psi(t)\vec{u}(t)} + g(t)$$

$$\vec{u}'(t) = \Psi^{-1}(t)\vec{g}(t)$$

$$\vec{u}(t) = \int \Psi^{-1}(t)\vec{g}(t)\,dt + \vec{c}$$

$$\boxed{\vec{x}(t) = \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt}$$

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Ex: $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = A\vec{x} + \vec{g}(t)$

$\lambda_1 = -3 \quad \xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \lambda_2 = -1, \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}, \qquad \Psi^{-1}(t) = \frac{1}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix}$

The DE for the coefficient $\vec{u}$ is:

$$\vec{u}'(t) = \Psi^{-1}(t)\vec{g}(t) = \frac{1}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ t \end{pmatrix} = \begin{pmatrix} e^{2t} - \frac{3}{2} te^{3t} \\ 1 + \frac{3}{2} te^{t} \end{pmatrix}$$

The DE's for the two components of $\vec{u}(t)$ and their solutions are

$$u_1' = e^{2t} - \frac{3}{2} te^{3t} \Rightarrow u_1(t) = \frac{1}{2} e^{2t} - \frac{1}{2} te^{3t} + \frac{1}{6} e^{3t} + c_1$$

$$u_2' = 1 + \frac{3}{2} te^{3t} \Rightarrow u_2(t) = t + \frac{3}{2} te^{t} - \frac{3}{2} e^{t} + c_2$$

Putting everything together

$$\vec{x}(t) = \Psi(t)\vec{u}(t)$$

we obtain

$$\vec{x}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + te^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} +$$

$$+ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

DIFERENTIAL EQUATIONS, DIAGONALIZATION

Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = (\lambda - 1)(\lambda + 1)$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Next we find the eigenvectors

$$\lambda_1 = 1, \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \xi_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The matrix of eigenvectors and its inverse are

$$T = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \quad T^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

After the cahnge of variables $\mathbf{x} = T\mathbf{y}$

$$\mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$$

The equations for the components of y and their soultions are

$$y_1' = y_1 + \frac{e^{2t}}{2} - \frac{1}{2} \Rightarrow y_1 = \frac{e^{2t}}{2} + \frac{1}{2} + c_1 e^t$$

$$y_2' = -y_2 - \frac{e^{2t}}{2} + \frac{3}{2} \Rightarrow y_2 = -\frac{e^{2t}}{6} + \frac{3}{2} + c_2 e^{-t}$$

The solution in the original variables is given by

$$\mathbf{x}(t) = T\mathbf{y}(t) = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{e^{2t}}{2} + \frac{1}{2} + c_1 e^t \\ y_2 = -\frac{e^{2t}}{6} + \frac{3}{2} + c_2 e^{-t} \end{pmatrix}$$

$$\mathbf{x}(t) = \begin{pmatrix} 3c_1 e^t + c_2 e^{-t} + \frac{4}{3} e^{2t} + 3 \\ c_1 e^t + c_2 e^{-t} + \frac{1}{3} e^{2t} + 2 \end{pmatrix}$$

The initial conditions settle the values of c_1 and c_2

$$\mathbf{x}(0) = \begin{pmatrix} 3c_1 + c_2 + \frac{4}{3} + 3 \\ c_1 + c_2 + \frac{1}{3} + 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \ c_1 = -\frac{3}{2} \ \ c_2 = -\frac{5}{6}$$

Substituting these values we have the solution of the IVP

$$\mathbf{x}(t) = \begin{pmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{pmatrix}$$

DIFERENTIAL EQUATIONS, VARIATION OF PARAMETERS

Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = (\lambda - 1)(\lambda + 1)$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Next we find the eigenvectors

$$\lambda_1 = 1, \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \xi_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A fundamental matrix is

$$\Psi(t) = \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}$$

Its inverse is

$$\Psi^{-1}(t) = \frac{1}{2} \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix}$$

The solution is given by

$$\mathbf{x}(t) = \Psi(t)\Psi^{-1}(0)\mathbf{x}(0) + \Psi(t) \int_0^t \Psi^{-1}(s)\mathbf{g}(s)\, ds$$

$$= \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \int_0^t \frac{1}{2} \begin{pmatrix} e^{-s} & -e^{-s} \\ -e^s & 3e^s \end{pmatrix} \begin{pmatrix} e^{2s} \\ 1 \end{pmatrix} ds$$

$$= \frac{1}{2} \begin{pmatrix} -3e^t + e^{-t} \\ -e^t + e^{-t} \end{pmatrix} + \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1 \\ \frac{3}{2}e^t - \frac{1}{6}e^{3t} - \frac{4}{3} \end{pmatrix}$$

$$\mathbf{x}(t) = \begin{pmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{pmatrix}$$