Diff Eq - §26 - Laplace Transform

Laplace transform: introduced by Laplace, but developed by Heaviside.

Def: Let $f(t)$, $t \geq 0$ be given. Its Laplace transform is the function $F(s)$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

whenever this improper integral converges.

Ex: Let $f(t) = e^{at}$, $t \geq 0$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \quad s > a.$$

rem: To explore fully the power of Laplace transform we need s to be a complex variable.

Th: Suppose that:

i) $f(t)$ is picewise continuous on the interval $0 \leq t \leq A$.

ii) $|f(t)| \leq k e^{at}$, $t \geq M$; $k, M, a \in \mathbb{R}$

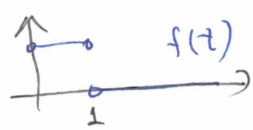
The Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$. ☒

rem: Functions which obey the condition ii) of this Theorem are said to be of exponential order.

Ex: $f(t) = t^t$ is not; neither is $g(t) = e^{t^2}$.

Ex: $f(t) = 1$. $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$

Ex:
$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$$



Value at 1 does not matter.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_0^1 = \frac{1-e^{-s}}{s}, \quad s > 0$$

Ex: $f(t) = \sin at$, $t > 0$. Then

$$F(s) = \mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt = \lim_{M\to\infty}\left[-\frac{e^{-st}\cos at}{a}\Big|_0^M - \frac{s}{a}\int_0^M e^{-st}\cos at\,dt\right.$$

$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt, \quad s > 0. \quad (*)$$

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$$\int_0^\infty e^{-st}\cos at\,dt = \lim_{M\to\infty}\left[\frac{e^{-st}\sin at}{a}\Big|_0^M + \frac{s}{a}\int_0^M e^{-st}\cos at\right]$$

$$= \frac{s}{a}F(s)$$

substituting back in (*):

$$F(s) = \frac{1}{a} - \frac{s}{a}\cdot\frac{s}{a}F(s), \quad F(s)\left(1+\frac{s^2}{a^2}\right) = \frac{1}{a}$$

$$F(s) = \frac{a}{s^2+a^2}, \quad s>0.$$

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rem: The Laplace transform is a linear operator.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}\{c_1 f_1(t) + c_2 f_2(t)\} =$$

$$= c_1\int_0^\infty e^{-st} f_1(t)\,dt + c_2\int_0^\infty e^{-st} f_2(t)\,dt =$$

$$= c_1\,\mathcal{L}\{f_1(t)\} + c_2\{f_2(t)\}.$$

ex: $f(t) = 4e^{-3t} - 5\sin 2t, \quad t>0$

$$\mathcal{L}\{f(t)\} = 4\,\mathcal{L}\{e^{-3t}\} - 5\,\mathcal{L}\{\sin 2t\} =$$

$$= 4\cdot\frac{1}{s+3} - 5\cdot\frac{2}{s^2+4}, \quad \left.\begin{matrix} s>-3 \\ s>0 \end{matrix}\right\} \Rightarrow s>0.$$