Diff Eq - §27 - Using Laplace Transform to solve initial value problems

- The Laplace transform of f' is related in a simple way to the Laplace transform of f.

Th: Suppose that f is continuous and f' is piecewise continuous on an interval $0 \le t \le A$. Suppose further that f is of exponential order, $\exists$ constants k, a, M s.t. $|f(t)| \le ke^{at}$, $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover,

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$$

Pr: We will prove the Th in the case f' is continuous. (In general see book)

$$\int_0^A e^{-st} f'(t)\,dt = \underbrace{e^{-st} f(t)\Big|_0^A}_{e^{-sA} f(A) - f(0)} + s\int_0^A e^{-st} f(t)\,dt$$

Take $A \to \infty$, $\quad \cancel{e^{-sA} f(A)} \to 0$ as $A \to \infty$

$$\mathcal{L}\{f'(t)\} = -f(0) + s\,\mathcal{L}\{f(t)\}. \quad \boxtimes$$

rem: If f', f'' the same conditions as f, f' above

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Cor: Suppose that the functions $f, f', \dots, f^{(n-1)}$ are continuous and of exponential order. Suppose $f^{(n)}$ is piecewise continuous. Then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Ex: Solve $y''(t) + y(t) = t$, $y(0) = 1$ and $y'(0) = 2$.

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}(t)$$

$$s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) + \mathcal{L}\{y(t)\} = \frac{1}{s^2}$$

$$s^2Y(s) - s - 2 + Y(s) = 1/s^2$$

$$Y(s) = \frac{1}{1+s^2}\left\{\frac{1}{s^2} + s + 2\right\}, \quad Y(s) = \frac{1 + s^2(s+2)}{s^2(s+1)}$$

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$$\frac{1+s^2(s+2)}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1+s^2(s+2) = (A+C)s^3 + (B+D)s^2 + As + B$$

$\Rightarrow A=0,\ C=1,\ B=1,\ D=1.$

$$Y(s) = \frac{1}{s^2} + \frac{s+1}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = t + \cos t + \sin t.$ $\boxtimes$

In general: for a 2nd order DE with constant coefficients:

$$ay'' + by' + cy = f(t)$$

Taking the Laplace transform we have

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$$

$$Y(s) = \frac{(as+b)y(0) + ay'(0) + F(s)}{as^2+bs+c}.$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

rem: Laplace transform converts DE into an algebraic equation. Note however that finding the inverse Laplace transform of $Y(s)$ might be difficult. {The inverse Laplace transform is given by a complex integral}. On continuous functions the Laplace transform is a bijection between function spaces.

Ex: Find the solution of the initial value problem

$$y'' + y = \sin 2t, \quad y(0) = 2, \quad y'(0) = 1.$$

Sol: $s^2Y(s) - sy(0) - y'(0) + Y(s) = \dfrac{2}{s^2+4}$

$$(s^2+1)Y(s) = 2s + 1 + \frac{2}{s^2+4}$$

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$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$2s^3 + s^2 + 8s + 6 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

$$\left.\begin{array}{l} A+C=2 \\ 4A+C=8 \end{array}\right\} A=2,\ C=0 \qquad \left.\begin{array}{l} B+D=1 \\ 4B+D=6 \end{array}\right\} \Rightarrow B=\frac{5}{3},\ D=-\frac{2}{3}$$

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$