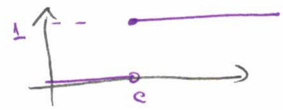
Diff Eq. - § 28 - Step functions

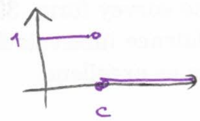
DE's with discontinuous forcing functions.

Def: The Heaviside function is

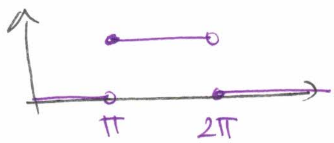
$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$



Ex: $y(t) = 1 - u_c(t)$,



Ex: $y(t) = u_{\pi}(t) - u_{2\pi}(t)$



(rectangular pulse).

constr. The Laplace transform u_c is

$$\mathcal{L}\{u_c(t)\} = \int_0^{\infty} e^{-st} u_c(t)\, dt = \int_c^{\infty} e^{-st}\, dt = \frac{e^{-cs}}{s}, \quad s > 0.$$

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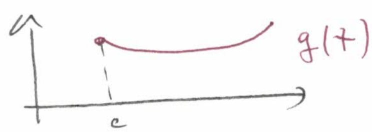
constr: For a given function f defined for $t \geq 0$ define its translation $g(t)$ via

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}, \quad \text{i.e. } g(t) = u_c(t) f(t-c)$$



Th: If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if $c > 0$, Then

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s), \quad s > a.$$

conversely,

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs} F(s)\}.$$

Pr: "Translation of $f(t)$ at a distance c corresponds to a multiplication of $F(s)$ by e^{-cs}"

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^{\infty} e^{-st} u_c(t) f(t-c)\, dt = \int_c^{\infty} e^{-st} f(t-c)\, dt = \qquad u = t-c$$

$$= \int_0^{\infty} e^{-(u+c)s} f(u)\, du = e^{-cs} \int_0^{\infty} e^{-su} f(u)\, du = e^{-cs} F(s). \quad \boxtimes$$

Ex: Find $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 0 & 0 \leq t \leq \pi/4 \\ \cos(t - \pi/4) & t \geq \pi/4 \end{cases}$

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$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_{\pi/4}(t)\cos(t-\pi/4)\} = e^{-\pi s/4}\mathcal{L}\{\cos t\} = e^{-\pi s/4}\frac{s}{s^2+1} \quad \boxtimes$$

Ex: Determine $\mathcal{L}^{-1}\{F(s)\}$, where $F(s) = \dfrac{1-e^{-2s}}{s^2}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = t - u_2(t)(t-2) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2. \end{cases} \quad \boxtimes$$

Th: If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$ Then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a+c.$$

Conversely, $e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}$.

Pr: $\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)\,d(t) = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c) \quad \boxtimes$

Ex: $F(s) = \dfrac{1}{s^2-4s+5}$ $\qquad \mathcal{L}^{-1}\{F(s)\} = ?$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} = e^{2t}\sin t. \quad \boxtimes$$