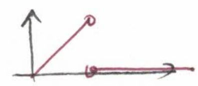
Diff Eq - § 29 - DE with discontinuous forcing functions

(1)

Ex: $y''(t) + y(t) = r(t)$, $t \geq 0$; $y(0) = 1$, $y'(0) = 2$, where

$$r(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases} = [1 - u_1(t)]\, t$$



solution:

$$R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{[1-u_1(t)]t\} = \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)\,t\} =$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)[(t-1)+1]\} = \frac{1}{s^2} - e^{-s}\mathcal{L}\{t\} - \mathcal{L}\{u_1(t)\} =$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$s^2 Y(s) - s - 2 + Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1 + s^2(s+2)}{s^2(s^2+1)} - e^{-s}\frac{1}{s^2(s^2+1)} - e^{-s}\frac{1}{s(s^2+1)}$$

From an example in lecture 31 : $\mathcal{L}^{-1}\left\{\frac{1+s^2(s+2)}{s^2(s^2+1)}\right\} = t + \cos t + \sin t$

Now notice that $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$ and hence

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t \quad \Rightarrow$$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2(s^2+1)}\right\} = u_1(t)\, f(t-1) = u_1(t)\left[(t-1) - \sin(t-1)\right]$$

Also $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$, $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t \Rightarrow$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s(s^2+1)}\right\} = u_1(t)\left[1 - \cos(t-1)\right]$$

$$y(t) = t + \cos t + \sin t + u_1(t)\left[-(t-1) + \sin(t-1) - 1 + \cos(t-1)\right]$$

$$y(t) = t + \cos t + \sin t + u_1(t)\left[-t + \sin(t-1) + \cos(t-1)\right]$$

$$y(t) = \begin{cases} t + \cos t + \sin t & 0 \leq t < 1 \\ \cos t + \sin t + \cos(t-1) + \sin(t-1) & t \geq 1 \end{cases}$$

Notice that this function is continuous at $t = 1$ (as is the derivative $y'(t)$). But the second derivative is discontinuous.

Diff Eq - §29 - DE with discontinuous forcing functions

Ex. $2y'' + y' + 2y = g(t)$, $y(0)=0$, $y'(0)=0$ where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & 0 \le t < 5 \text{ and } t \ge 20. \end{cases}$$

Sol: $2s^2Y(s) - 2s\,y(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \frac{1}{s}e^{-5s} - \frac{1}{s}e^{-20s}$

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}$$

$$\frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2} \qquad A = 1/2,\ B = -1,\ C = -1/2$$

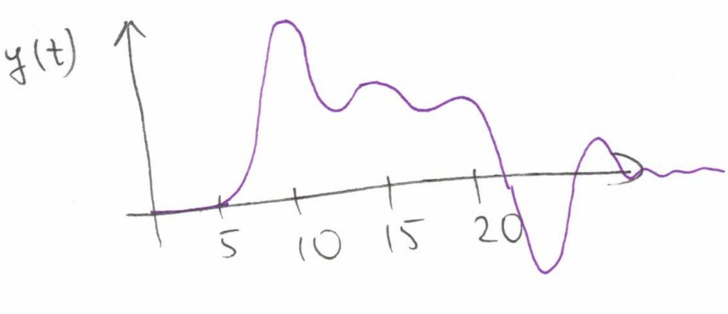
$$\frac{1}{s(2s^2+s+2)} = \frac{1/2}{s} - \frac{s+1/2}{2s^2+s+2} = \frac{1/2}{s} - \frac{1}{2}\left\{\frac{s+1/4}{(s+1/4)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} + \frac{1/4}{(s+1/4)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(2s^2+s+2)s}\right\} \underset{\uparrow\, 9,10}{=} \frac{1}{2} - \frac{1}{2}\left\{e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) + \frac{1}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\left(e^{-5s} - e^{20s}\right)\frac{1}{s(2s^2+s+2)}\right\} \underset{\uparrow\, 13}{=}$$

$$= u_5(t)\left\{\frac{1}{2} - \frac{1}{2}\left\{e^{-(t-5)/4}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) + \frac{1}{\sqrt{15}}e^{-(t-5)/4}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right\}\right\}$$

$$- u_{20}(t)\left\{\frac{1}{2} - \frac{1}{2}\left\{e^{-(t-20)/4}\cos\left(\frac{\sqrt{15}}{4}(t-20)\right) + \frac{1}{\sqrt{15}}e^{-(t-20)/4}\sin\left(\frac{\sqrt{15}}{4}(t-20)\right)\right\}\right\}.$$



Again $y(t), y'(t)$ are continuous, but $y''(t)$ is discontinuous at $t=5$, $t=20$.