Diff Eq - §3 - Linear Equations.
Method of integrating factors.

Here we consider 1st order linear DE's of the form:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t), \qquad y = y(t).$$

Sometimes we can solve by simply integrating the equation:

Ex: $(4+t^2)\frac{dy}{dt} + 2ty = 4t$

Notice that $(4+t^2)\frac{dy}{dt} + 2ty = \frac{d}{dt}\left[(4+t^2)y\right]$

$$\int\left\{\frac{d}{dt}\left[(4+t^2)y\right]\right\}dt = \int 4t\,dt$$

$$(4+t^2)y = 2t^2 + c \qquad y = \frac{2t^2+c}{4+t^2}$$

Usually the terms are not going to be this nicely matching. But we can multiply the DE by a function $\mu(t)$ called integrating factor to try to arrange that the LHS is a derivative $d/dt\{f(t)y(t)\}$.

Ex: $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}, \qquad y(0) = 1.$

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3}$$

Since we have $\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \mu'(t)y$

we must have $\mu'(t) = \frac{1}{2}\mu(t)$ i.e. $\frac{d\mu}{dt} = \frac{1}{2}\mu$

$$\frac{d\mu}{\mu} = \frac{1}{2}dt \qquad \ln|\mu| = \frac{1}{2}t + d, \quad |\mu| = e^d e^{1/2 t}, \quad \mu = ce^{t/2}$$

We have an integrating factor $\forall c \neq 0$. Select $\mu(t) = e^{t/2}$

$$e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6}, \qquad \frac{d}{dt}\left(e^{t/2}y\right) = \frac{1}{2}e^{5t/6}$$

$$e^{t/2}y = \frac{3}{5}e^{5t/6} + c, \quad y = \frac{3}{5}e^{t/3} + ce^{-t/2}; \quad y(0) = 1, \quad \boxed{y = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}}$$

Diff Eq - §3 - Method of integrating factors

Gen Ex: $\frac{dy}{dt} + ay = g(t)$, $a = \text{const}$.

$\mu(t)\frac{dy}{dt} + a\mu y = \mu g$ $\quad \frac{d\mu}{dt} = a\mu \quad \mu = ce^{at} \to \mu = e^{at}$

$e^{at}\frac{dy}{dt} + a e^{at} y = e^{at} g(t) \quad \frac{d}{dt}\left[e^{at} y\right] = e^{at} g(t)$

$e^{at} y = \int e^{at} g(t)\,dt + c$

$$\boxed{y = e^{-at}\int e^{at} g(t)\,dt + ce^{-at}}$$

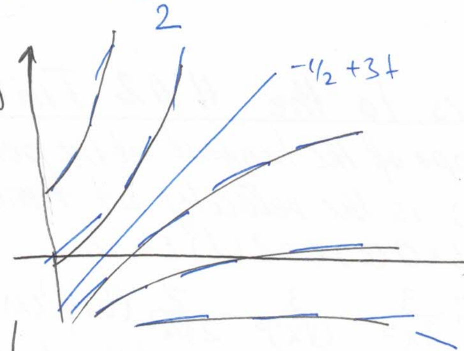
Stud. Ex: $\frac{dy}{dt} - 2y = 4 - 6t$; $a = -2$ so the integrating factor is $\mu(t) = e^{-2t}$

$e^{-2t}\frac{dy}{dt} - 2e^{-2t} y = 4e^{-2t} - 6te^{-2t}$

$\int \frac{d}{dt}\left[e^{-2t} y\right] dt = \int \left(4e^{-2t} - 6te^{-2t}\right) dt$.

$e^{-2t} y = -2e^{-2t} + 3te^{-2t} + \frac{3}{2}e^{-2t} + c$

$y = -\frac{1}{2} + 3t + ce^{2t}$



The boundary between the solutions which go to $+\infty$ and the ones which go to $-\infty$ is $c = 0$, $y = -1/2 + 3t$

Return to the general first order linear equation:

$\frac{dy}{dt} + p(t) y = g(t)$ $\leftarrow$ integrating factor $\mu(t)$

$\underbrace{\mu(t)\frac{dy}{dt} + \mu(t) p(t) y}_{\frac{d}{dt}\left[\mu(t) y\right]} = \mu(t) g(t)$

$\Rightarrow \quad \frac{d\mu}{dt} = \mu(t) p(t)$

$\int \frac{d\mu}{\mu} = \int p\,dt$, $\ln(\mu(t)) = \int p(t)\,dt + k$, say $k = 0$

$$\mu(t) = \exp \int p(t)\,dt$$

Returning to the equation we have

$$\frac{d}{dt}\left[\mu(t) y\right] = \mu(t) g(t)$$

$$\mu(t) y = \int \mu(t) g(t)\,dt + c$$

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s) g(s)\,ds + c\right]$$ ← general solution.

Ex: Solve $\frac{dy}{dt} + \frac{2}{t} y = t - 1$

$$\mu(t) = \exp \int \frac{2}{t}\,dt = \exp(2 \ln t) = t^2$$

$$t^2 \frac{dy}{dt} + 2ty = t^3 - t^2$$

$\frac{d}{dt}\left[t^2 y\right] = t^3 - t^2$, $t^2 y = \int (t^3 - t^2)\,dt$

$t^2 y = t^4/4 - t^3/3 + c$ $\quad y = t^2/4 - t/3 + c/t^2$.

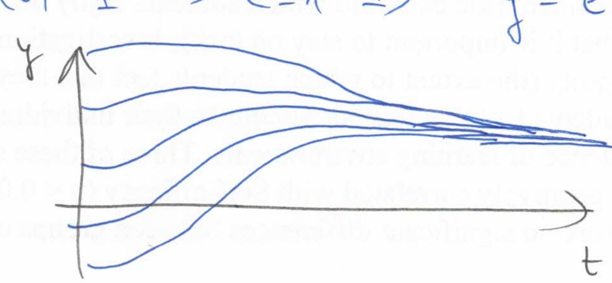
Ex: Solve $2y' + ty = 2$, $y(0) = 1$

$y' + \frac{t}{2} y = 1$, $p(t) = t/2$ $\quad \mu(t) = \exp\left(t^2/4\right)$

$$e^{t^2/4} y' + \frac{t}{2} e^{t^2/4} y = e^{t^2/4}$$

$e^{t^2/4} y = \int e^{t^2/4}\,dt + C$ i.e. $e^{t^2/4} y = \int_0^t e^{s^2/4}\,ds + C$

$y(0) = 1 \Rightarrow C = 1 \Rightarrow y = e^{-t^2/4} \int_0^t e^{s^2/4}\,ds + e^{-t^2/4}$.



Few integral curves.