Diff Eq - § 30 - Impulse functions

Consider the case where the driving force $g(t)$ is large, but for a very short period of time in a DE such as

$$ay'' + by' + cy = g(t).$$

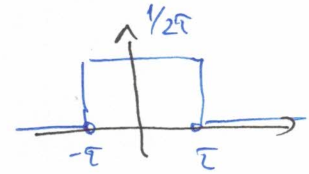
Say $g(t)$ is large on $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$ and is otherwise zero. In a mechanical system, where $g(t)$ is a force the integral

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$

is the total impulse of the force $g(t)$. (change in momentum).

WLOG suppose $t_0 = 0$. Also suppose that

$$g(t) = d_\tau(t) = \begin{cases} 1/2\tau & -\tau < t < \tau \\ 0 & \text{elsewhere.} \end{cases}$$



Now $I(\tau) = 1$ for each $\tau \neq 0$. Moreover $\lim_{\tau \to 0^+} I(\tau) = 1$. But the limit $\lim_{\tau \to 0} d_\tau(t)$ does not exist in the space of functions. Instead this limit defines a distribution (generalized function) known as the Dirac delta function: $\delta(t)$. The Dirac delta function has the following properties

$$\delta(t - t_0) = 0 \qquad t \neq t_0$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1.$$

The Laplace transform of the distribution $\delta(t - t_0)$ is well defined. We will compute it via

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\tau \to 0^+} \mathcal{L}\{d_\tau(t - t_0)\}$$

Now $\mathcal{L}\{d_\tau(t - t_0)\} = \int_0^\infty e^{-st} d_\tau(t - t_0)\,dt = \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t - t_0)\,dt$

$$= \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st}\,dt = \frac{1}{2s\tau} e^{-st_0}\left(e^{s\tau} - e^{-s\tau}\right) = e^{-st_0} \frac{\sinh(s\tau)}{s\tau}.$$

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Applying L'Hospital Rule

$$\lim_{\tau \to 0^+} \frac{\sinh s\tau}{s\tau} = \lim_{\tau \to 0^+} \frac{s \cosh s\tau}{s} = 1.$$

Cor: $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$

rem: Actually This result is obvious if we take into consideration The defining property of The Dirac delta function

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

Ex:  A mass attached to spring is released 1 m from its equilibrium position and begins to vibrate. After π seconds, The mass is struck by a hammer. The dynamics is modelled by

$$x'' + 9x = 3\delta(t-\pi), \quad x(0) = 1, \; x'(0) = 0.$$

Determine $x(t)$.

sol: $\mathcal{L}\{x''+9x\} = \mathcal{L}\{3\delta(t-\pi)\} \Rightarrow s^2X(s) - s + 9X(s) = 3e^{-\pi s}$

$$X(s) = \frac{s}{s^2+9} + e^{-\pi s}\frac{3}{s^2+9}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{3}{s^2+9}\right\} =$$

$$= \cos 3t + u_\pi(t)\sin(3(t-\pi))$$

$$= \begin{cases} \cos 3t, & t < \pi \\ \cos 3t - \sin 3t, & t \geq \pi \end{cases} = \begin{cases} \cos 3t & t < \pi \\ \sqrt{2}\cos\left(3t + \frac{\pi}{4}\right) & t \geq \pi \end{cases}$$