Diff Eq - §31 - Stability

We will restrict our attention to autonomous systems of DE's and just so that we can draw pretty pictures we will mostly consider the 2x2 case. So we will be concerned with systems of the form

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y)$$

with initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$. The system is autonomous since the functions $F(x,y)$, $G(x,y)$ do not depend on t.

We will also write this type of system in vector form

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}), \quad \vec{x}(t_0) = \vec{x}^{\,0}, \quad \text{where } \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

A solution around the initial point $\vec{x}^{\,0}$ exists if $F(x,y)$, $G(x,y)$ are continuous with continuous partial derivatives and the solution can be viewed as a curve in the xy-plane.

Ex: The linear system $\vec{x}' = A\vec{x}$ with a constant coefficient matrix A is autonomous.

rem: After the dichotomy linear/nonlinear the next useful dichotomy is autonomous/nonautonomous. e.g. for autonomous systems the phase diagram is indepent of t: a single phase portrait displays information about the solutions of the system regardless of the initial conditions.

Def: Consider an autonomous system of the form $\vec{x}' = f(\vec{x})$.

a) The points where $\vec{x}' = f(\vec{x}) = 0$ are called critical points. (These correspond to constant (equilibrium) solutions.

b) A critical point $\vec{x}^{\,c}$ is said to be stable if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. the solutions with initial conditions $\vec{x}^{\,0} = \vec{x}(0)$ satisfy

$$\|\vec{x}(0) - \vec{x}^{\,c}\| < \delta \Rightarrow \|\vec{x}(t) - \vec{x}^{\,c}\| < \varepsilon, \quad t \in [0, \infty).$$

Informally this condition says that every solution which starts sufficiently close to $\vec{x}^c$ will stay within a distance ε to $\vec{x}^c$ for every ε and for every $t \in [0, \infty)$.

A critical point which is not stable is **unstable.**

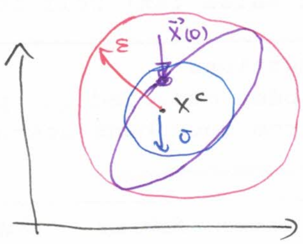
c) A critical point $\vec{x}^c$ is said to be **asymptotically stable** if it is **stable** and $\exists \delta_0 > 0$ such that for initial conditions

$$\| \vec{x}(0) - \vec{x}^c \| < \delta_0$$

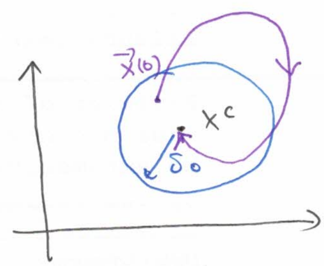
The solutions satisfy

$$\lim_{t \to \infty} \vec{x}(t) = \vec{x}^c$$

Pictures:



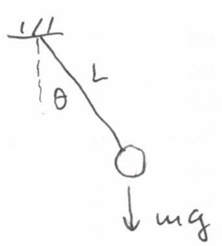
Stable.



Asymptotically stable.

rem: The limiting condition $\lim_{t \to \infty} \vec{x}(t) = \vec{x}^c$ does not imply stability. Examples exist where trajectories start close to $\vec{x}^c$, but then move arbitrarily far away before returning to $\vec{x}^c$ as $t \to \infty$.

Ex: Pendulum. The rate of change of angular momentum is equal to the moment of the total force:



$$mL^2 \frac{d^2\theta}{dt} = -\Gamma L \frac{d\theta}{dt} - mgL \sin\theta \qquad \Gamma\text{-friction}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$

In 1^{st} order form:

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

The critical points are $y = 0,\ x = 0$ - stable; $y = 0,\ x = \pi$ - unstable

If $\Gamma \neq 0$, $(0,0)$ is asymptotically stable. If $\Gamma = 0$, $(0,0)$ is stable but not asymptotically stable.

Ex: Consider the system

$$\frac{dx}{dt} = -(x-y)(1-x-y)\,; \qquad \frac{dy}{dt} = x(2+y)$$

The critical points are found by solving

$$(x-y)(1-x-y) = 0, \quad x(2+y) = 0.$$

There are four critical points $(0,0)$, $(0,1)$, $(-2,-2)$, $(3,-2)$.

$(0,0)$ is a saddle point ; $(0,1)$ is a spiral attractor

$(-2,-2)$ is an attractor node ; $(3,-2)$ is a a repeller node.

→ see phase diagrams in the appendix

1. The saddle point $(0,0)$ is unstable
2. The spiral attractor $(0,1)$ is asymptotically stable.
3. The attractor node $(-2,-2)$ is asymptotically stable
4. The repeller node $(3,-2)$ is unstable.

Def: Let P be a point in the xy-plane so that a trajectory through P approaches a critical point as $t \to \infty$. We say this trajectory is attracted by the critical point. The set of all such points P is the basin of attraction of the critical point. A trajectory which bounds a basin of attraction is called a separatrix.

rem: Determination of the basins of attraction is important for the understanding of the large-scale behavior of an autonomous system.

The trajectories of a 2d autonomous system

$$dx/dt = F(x,y), \qquad dy/dt = G(x,y)$$

can sometimes be determined by solving

$$dy/dx = G(x,y)/F(x,y)$$

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Ex: Find the trajectories of the system

$$x' = 4 - 2y, \quad y' = 12 - 3x^2$$

Notice first that $(-2,2)$ and $(2,2)$ are critical points. The trajectories are obtained by solving

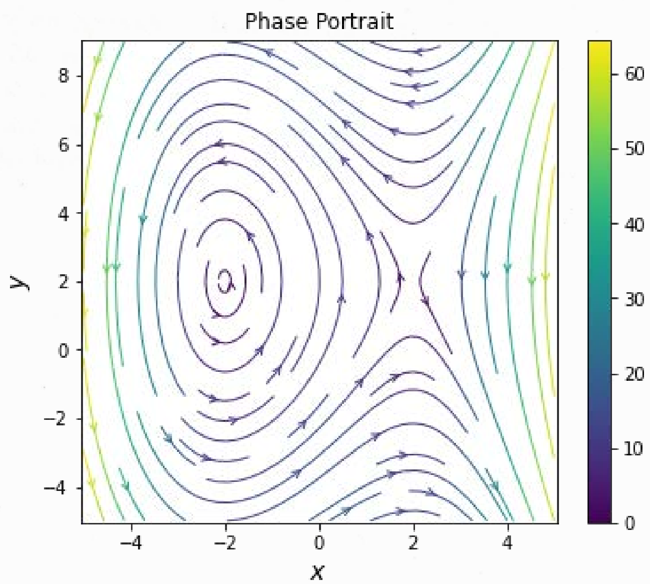
$$\frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}.$$

The solutions are

$$4y - y^2 - 12x + x^3 = C.$$

→ see phase diagram in the appendix

$(-2,2)$ is a center (stable, but not asymptotically stable).
$(2,2)$ is a saddle point unstable.



Phase portraits for the example on p. 3