Diff Eq - § 32 - Locally linear systems

Th: Consider the linear system $\vec{x}' = A x$, with $A = \text{const.}$, $\det A \neq 0$.
The critical point $\vec{x} = 0$ of this linear system is
- asymptotically stable if the eigenvalues $\lambda_1, \lambda_2 < 0$ or $\mathrm{Re}\lambda_1$, $\mathrm{Re}\lambda_2 < 0$
- stable if λ_1, λ_2 are purely imaginary
- unstable if either of λ_1, λ_2 is > 0 or has a positive real part

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Small perturbations.

Most sensitive to perturbations are the stable critical points where $\lambda_1 = i\mu$, $\lambda_2 = -i\mu$. If a slight changes in the matrix A are made, the eigenvalues will become $\lambda_1' = \lambda' + i\mu'$, $\lambda_2' = \lambda' - i\mu'$ with $\lambda' \sim 0$, $\mu' \sim \mu$. Generically $\lambda' \neq 0$ and if $\lambda' < 0$ the critical point is asymptotically stable and if $\lambda' > 0$ it is unstable.

Linear approximations to nonlinear systems.

Consider the nonlinear autonomous system

$$\vec{x}' = f(\vec{x})$$

Let $\vec{x}^0$ be a critical point $f(\vec{x}^0) = \vec{0}$. WLOG we can assume $\vec{x}^0 = \vec{0}$ since we can always implement a translation to $\vec{u} = \vec{x} - \vec{x}^0$.

So let $\vec{x}^0 = \vec{0}$ be an isolated critical point and write the system as

$$\vec{x}' = \underbrace{A\vec{x}}_{\text{linear term}} + \underbrace{\vec{g}(\vec{x})}_{\text{terms quadratic and higher order}}$$ ← Locally linear system

Ex:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2 - xy \\ -0.75xy - 0.25y^2 \end{pmatrix}$$

Ex: Pendulum: $x' = y$, $y' = -\omega^2 \sin x - \gamma y$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \omega^2 \begin{pmatrix} 0 \\ \sin x - x \end{pmatrix}$$

Notice $\sin x - x = -x^3/3! + \dots$

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Consider the autonomous system

$$x' = F(x,y), \quad y' = G(x,y)$$

with critical point (x_0,y_0). Assume the functions $F(x,y), G(x,y)$ have Taylor expansions about the critical point:

$$F(x,y) = \cancel{F(x_0,y_0)}^{0} + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) + \eta_1(x,y)$$

$$G(x,y) = \cancel{G(x_0,y_0)}^{0} + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0) + \eta_2(x,y)$$

Then the system can be written as

$$\frac{d}{dt}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = \begin{pmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \begin{pmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{pmatrix}$$

The matrix

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}(x_0,y_0)$$

is called the Jacobian matrix.

A linear approximation near the critical point (x_0,y_0) of the nonlinear system above is

$$\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}' = F\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

Ex: For the pendulum $F(x,y) = y$, $G(x,y) = -\omega^2 \sin x - \gamma y$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$J = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \Rightarrow \text{locally } \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\gamma-\lambda \end{pmatrix} = \lambda^2 + \gamma\lambda + \omega^2 = 0 \qquad \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}$$ Stable.

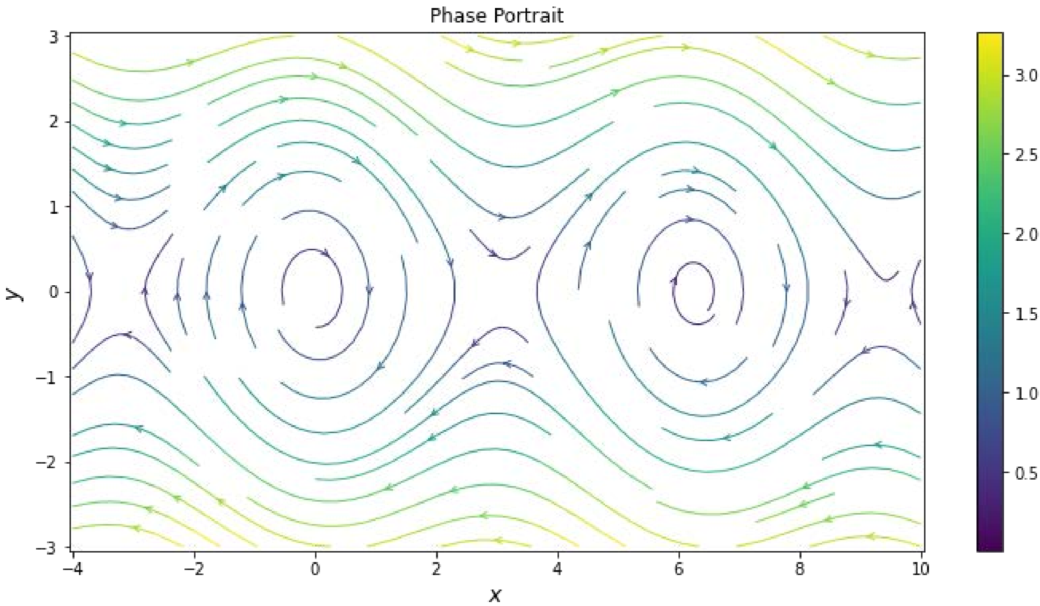
But there is another critical point $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$. The Jacobian at this point is:

$$J = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}$$

$$\det\begin{pmatrix} -\lambda & 1 \\ \omega^2 & -\gamma-\lambda \end{pmatrix} = \lambda^2 + \gamma\lambda - \omega^2 = 0 \qquad \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}$$

Unstable!

see Appendix!



Phase portrait for the pendulum.