Diff Eq - § 34 - Predator-prey models

x - prey ; y - predator

Lotka - Volterra equations (1925 - 1926).

$$\frac{dx}{dt} = ax - \alpha xy = x(a - \alpha y)$$

$$\frac{dy}{dt} = -cy + \gamma xy = y(-c + \gamma x)$$

a - growth rate of prey ; c - death rate of predator

α, γ - interactions between the two species.

ex: $x' = x(1 - 0.5y) = F(x,y)$; $y' = y(-0.75 + 0.25x) = G(x,y)$

The critical points are solutions of

$$x(1-0.5y) = 0 \quad ; \quad y(-0.75 + 0.25x) = 0.$$

The are two critical points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

The Jacobian is:

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$$

At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ unstable saddle point.

At $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$: $\begin{pmatrix} x-3 \\ y-2 \end{pmatrix}' = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} x-3 \\ y-2 \end{pmatrix}$

$$\lambda_{1,2} = \pm \frac{\sqrt{3}}{2} i \quad \Rightarrow \text{ stable critical point.}$$

Q? Does this stability survive the nonlinear perturbation? Let's find expressions for the trajectories.

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{y(-0.75 + 0.25x)}{x(1 - 0.5y)}$$

$$\frac{1 - 0.5y}{y} dy = \frac{-0.75 + 0.25x}{x} dx$$

$$\ln y - 0.5y + 0.75 \ln x - 0.25x = C$$

These are closed curves (see Appendix). $x(t), y(t)$ are periodic.

The predator population lags behind the prey.

Diff Eq - §34 - Predator-prey models

The general Lotka-Volterra system can be analyzed the same way:

$$x' = x(a - \alpha y), \quad y' = y(-c + \gamma x)$$

The critical points are $\begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}c/\gamma\\a/\alpha\end{pmatrix}$. The Jacobian is

$$J = \begin{pmatrix} a - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{pmatrix}$$

At $\begin{pmatrix}0\\0\end{pmatrix}$: $J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ - This is a saddle point (unstable).

At $\begin{pmatrix}c/\gamma\\a/\alpha\end{pmatrix}$: $J = \begin{pmatrix} 0 & -\alpha c/\gamma \\ \gamma a/\alpha & 0 \end{pmatrix}$. The eigenvalues are $\lambda_{1,2} = \pm i\sqrt{ac}$, so

This is a stable center.

We can solve for the trajectories

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-c+\gamma x)}{x(a-\alpha y)}$$

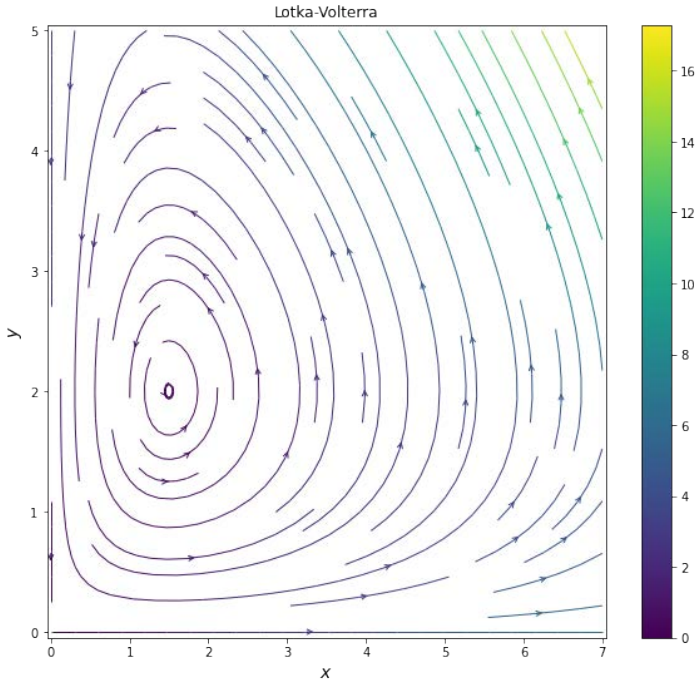
$$\Rightarrow a \ln y - \alpha y + c \ln x - \gamma x = C,$$

which are again closed curves. The predator/prey populations oscillate with the predator being about a quarter cycle behind.

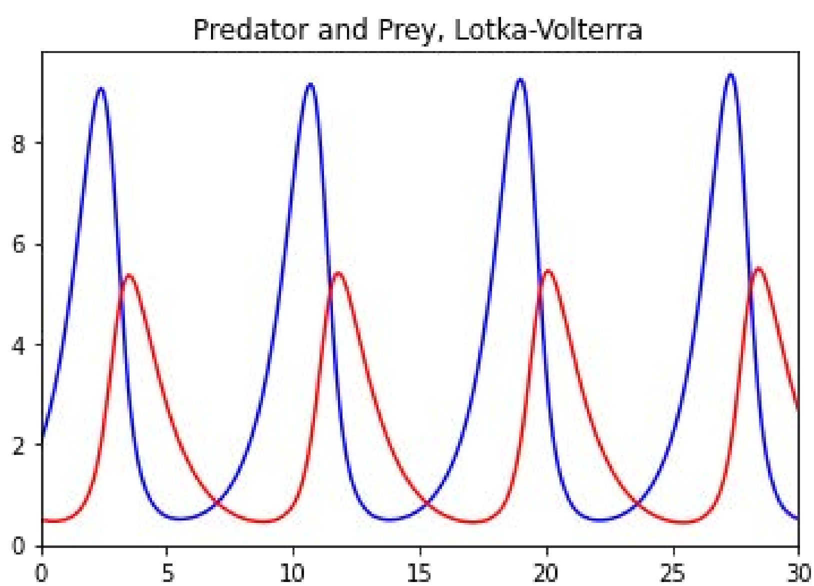
Ex: Hudson Bay company data 1845-1935: lynx & snowshoe hare. Periodicity of 10 years.

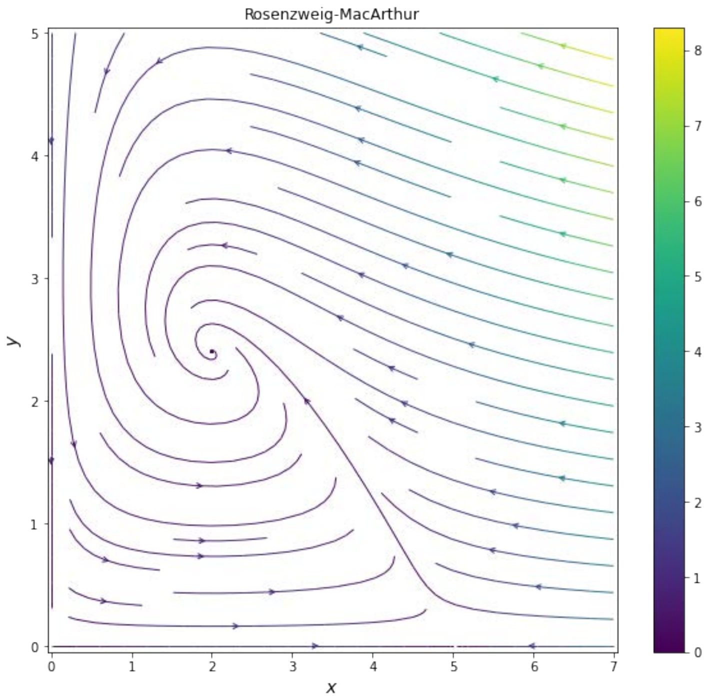
If the prey population is much larger than the predator population the interaction term is modified as in the Rosenzweig-MacArthur system:

$$x' = x\left(1 - 0.2x - \frac{2y}{x+6}\right) \qquad y' = y\left(-0.25 + \frac{x}{x+6}\right)$$



Lotka-Volterra system





Rosenzweig-MacArthur system