§ 35 - Chaos: The Lorenz equations

The Lorenz system arose in meteorology. If the temperature difference is small there is no significant motion of layers with different temperatures even if the hotter air is underneath cooler air. However, if ΔT is large enough the warmer air rises up giving rise to a convective flow. The Lorenz system describes this phenomena:

$$dx/dt = \sigma(-x+y)$$
$$dy/dt = rx - y - xz$$
$$dz/dt = -bz + xy$$

x is related to the flow, y, z are related to temperature variations in the horizontal and vertical directions.

For air $\sigma \approx 10$, $b \approx 8/3$. The parameter r is related to the temperature gradient.

The Jacobian of the system is

$$J = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ r-z & -1 & -x \\ y & x & -8/3 \end{pmatrix}$$

The critical points depend on the value of r. For $r \le 1$ the only critical point is $P_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. When $r > 1$ there are three critical points

$$P_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{pmatrix}$$

At $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $J = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}$ and the eigenvalues are

$$\lambda_1 = -\frac{8}{3}, \quad \lambda_2 = \frac{-11 - \sqrt{81+40r}}{2}, \quad \lambda_3 = \frac{-11 + \sqrt{81+40r}}{2}$$

For $r < 1$ all three eigenvalues are negative and P_1 is asymptotically stable. For $r > 1$, $\lambda_3 > 0$ and P_1 is unstable. $r = 1$ corresponds to

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the initiation of convex flow.

At the second critical point $P_2 = \left(\sqrt{8(r-1)/3}, \sqrt{8(r-1)/3}, r-1\right)$ for $r>1$ the Jacobian is

$$J = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8(r-1)/3} \\ \sqrt{8(r-1)/3} & \sqrt{8(r-1)/3} & -8/3 \end{pmatrix}$$

and the eigenvalues depend on r in the following way:

For $1 < r < r_1 \simeq 1.3456$ there are three negative real eigenvalues

For $r_1 < r < r_2 \cong 24.737$ there is one negative real eigenvalue and two complex eigenvalues with negative real part

For $r_2 < r$ there is one negative real eigenvalue and two complex eigenvalues with positive real part.

The results for the critical point P_3 are analogous.

Summary:

$0 < r < 1$, the only critical point is P_1 and it is asymptotically stable

$1 < r < r_1 \simeq 1.3456$, P_1 is unstable, P_2 and P_3 are asymptotically stable. All nearby solutions approach either P_2 or P_3 exponentially.

$r_1 < r < r_2 \simeq 24.737$, P_1 is unstable, P_2 and P_3 are asymptotically stable. Nearby solutions approach either P_2 or P_3 on inward spiral.

$r_2 < r$. All critical points are unstable. However all solutions remain bounded as $t \to \infty$. It can be shown that all solutions approach a limiting set of points that has zero volume.

The Lorenz system exhibits extreme sensitivity to the initial conditions $\to$ chaos (see Appendix). This sensitivity is quantified by the set of Lyapunov exponents, which characterize the stretching of shrinking of phase space is a set of (eigen-) directions.

§ 3.5 - Chaos in the Lorentz system

Consider a system of autonomous DE's

$$\vec{x}' = f(\vec{x})$$

Compute the Jacobian matrix $J_{ij}(t) = \left.\frac{d f_i(x)}{d x_j}\right|_{x(t)}$. The evolution of the tangent (linearization) vectors to the phase space is given by

$$\vec{Y}' = JY, \quad Y_{ij}(0) = \delta_{ij}.$$

The matrix Y describes how a small change at a point $\vec{x}(0)$ propagates to the point $\vec{x}(t)$. The limit

$$\Lambda = \lim_{t\to\infty} \frac{1}{2t} \log\left(Y(t)\, Y^T(t)\right)$$

defines a matrix. The Lyapunov exponents $\{\lambda_i\}$ are defined as the eigenvalues of Λ.

The spectrum of Lyapunov exponents depends on the starting point $\vec{x}(0)$. Most of the time we are interested in the set of attractors of a system of DE's and normally there is one set of exponents associated with each attractor.

- If $\vec{x}(0)$ is a critical point the Lyapunov exponents are the eigenvalues of $J(\vec{x}(0))$.
- For $\vec{x}(0)$ on a periodic orbit $\vec{x}(T) = \vec{x}(0)$ we need to integrate over the orbit
- Numerical approaches exist for aperiodic solutions.

Ex: The Lyapunov exponents for the Lorenz (strange) attractor are

$$\lambda_1 = 2.16, \quad \lambda_2 = 0, \quad \lambda_3 = -32.4$$

Link to a web page with a collection of strange attractors.

First component of the Lorenz system. Initial conditions (5,5,5) in blue and (5.01,5,5) in red.