Diff Eq - §5 - Modeling with 1st order DE's

DE model's relate variables and parameters in the problem:

→ make predictions

→ vary parameters (it would be impractical to do this physically)

(1) Construction of the model: identify variables; write equations summarizing the interactions between the variables.

(2) Solve: analytical, numerical solutions.

(3) Validate → compare with data, experiments, intuitions.

Ex: Mixing problem:

$Q(t)$ - amount of salt in the tank (kg)

$V(t)$ - volume of water in the tank (ℓ)

$c_i(t)$ - inflow salt concentration (kg/ℓ)

$c_o(t)$ - outflow salt concentration (kg/ℓ)

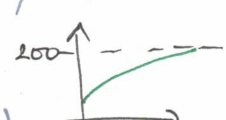
$r_i(t)$ - inflow rate (ℓ/min)

$r_o(t)$ - outflow rate (ℓ/min).

$$\frac{dQ}{dt} = \underbrace{r_i(t)c_i(t)}_{\text{Rate at which salt enters the tank}} - \underbrace{r_o(t)\frac{Q}{V(t)}}_{\text{Rate at which salt leaves the tank.}}$$

$$\frac{dV}{dt} = r_i(t) - r_o(t), \quad V(t) = V(0) + \int_0^t [r_i(s) - r_o(s)]\,ds.$$

e.g.: A tank has 1000 ℓ of water, in which 20 kg of salt is dissolved. A valve is opened and water containing 0.2 kg/ℓ of salt flows in the tank at a rate 5ℓ/min. The mixture in the tank is well stirred and drains at a rate of 5ℓ/min. Find $Q(t) = ?$. Find $\lim_{t\to\infty} Q(t) = ?$



Sol: $\frac{dQ}{dt} = 5(0.2) - 5\frac{Q}{1000} = 1 - \frac{Q}{200} = \frac{-1}{200}(-200 + Q)$

$\frac{dQ}{Q-200} = -\frac{1}{200}dt, \quad Q(t) = 200 + ke^{-t/200}, \quad Q(t) = 200 - 180e^{-t/200}.$

Ex: A lake initially contains 10 millon gal of fresh water. Water containing toxic chemicals flows into the lake at a rate of 5 million gal/year. The concentration of chemical in the inflow varies according to the expression $\gamma(t) = 2 + \sin 2t$. Construct a DE describing the amount of chemical in the lake. Find and interpret a solution.

Sol: $\frac{dQ}{dt}$ = rate in - rate out

rate in $= (5 \times 10^6)\,\frac{\text{gal}}{\text{year}}\,(2+\sin 2t)\,\frac{g}{\text{gal}} = 5\times 10^6 (2+\sin 2t)$

rate out $= (5\times 10^6)\,\frac{\text{gal}}{\text{year}}\;\frac{Q(t)}{10^7}\,\frac{g}{\text{gal}} = \frac{Q(t)}{2}$

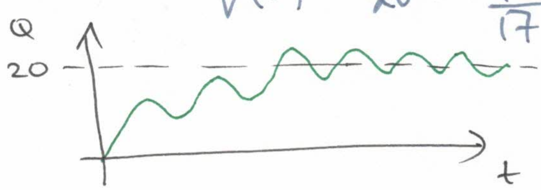
$\frac{dQ}{dt} = (5\times 10^6)(2+\sin 2t) - \frac{Q(t)}{2}$ Let $q(t) = \frac{Q(t)}{10^6}$. $q(t)$ is measured in millions of grams, i.e. tons.

$$\frac{dq}{dt} + \frac{1}{2} q = 10 + 5\sin 2t$$

Integrating factor $e^{t/2}$. Multiplying and integrating

$$q(t) = 20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t + c e^{-t/2} \qquad q(0)=0$$

$$q(t) = 20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t - \frac{300}{17} e^{-t/2}$$



Ex: Chemical reactions: Consider the reaction

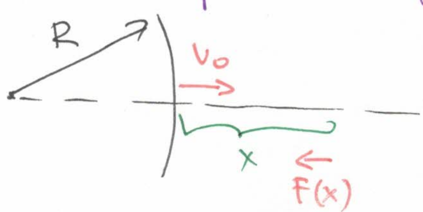
$$CH_3Cl + NaOH \rightarrow CH_3OH + NaCl$$

(methyl chloride) (Sodium hydroxide) (Methyl alcohol) (Salt)

One molecule of CH_3Cl combines with one molecule of $NaOH$. The rate at which the reaction proceeds is proportional to the product of the remaining concentrations of CH_3Cl and $NaOH$. If X denotes the amount of CH_3OH formed at time t and α, β the initial amounts $(t=0)$ of CH_3Cl and $NaOH$ respectively the rate of the reaction is given by

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X).$$

Ex: (Escape velocity) A body of mass m is shot straight up with initial velocity from the surface of Earth. Find a formula for the velocity as a function of the altitude x. Assume no friction, but take into account variable gravity.



$$F(x) = -mg\frac{R^2}{(R+x)^2} \qquad ma = m\frac{dv}{dt} = F = -mg\frac{R^2}{(R+x)^2}$$

But $\dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\dfrac{dv}{dx} \Rightarrow \boxed{v\dfrac{dv}{dx} = -\dfrac{gR^2}{(R+x)^2}}$ $\quad v(0) = v_0$

$$\int v\,dv = -gR^2\int\frac{dx}{(R+x)^2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{R+x} + c \qquad c = \frac{v_0^2}{2} - gR$$

$$v^2 = v_0^2 - 2gR + \frac{2gR^2}{R+x}, \quad v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}$$

Maximum altitude: $v=0$ $x = x_{max} \Rightarrow$

$$x_{max} = \frac{v_0^2 R}{2gR - v_0^2}$$

Escape velocity $x_{max} \to \infty$. $\Rightarrow$ $v_{esc} = \sqrt{2gR}$

On Earth: $v_{esc} = \sqrt{2\cdot(9.8)(6371\times10^3)} = 11175\, m/s = 11.175\, km/s.$

Solution of a DE.

$$\frac{dq}{dt} + \frac{1}{2}q = 10 + 5\sin 2t \qquad \times \mu(t)$$

$$\mu(t)\frac{dq}{dt} + \frac{1}{2}\mu(t)q(t) = \mu(t)(10 + 5\sin 2t)$$

$$\overset{\shortparallel}{\frac{d}{dt}[\mu(t)q]} \Rightarrow \quad \frac{d\mu}{dt} = \frac{1}{2}\mu(t) \quad \int\frac{d\mu}{\mu} = \int\frac{1}{2}dt \quad \mu = e^{1/2t}$$

$$\int (10 + 5\sin 2t)e^{1/2t} = \int 10e^{1/2t}dt + \int 5\sin 2t\, e^{1/2t}dt$$

$$= 10\,\frac{e^{1/2t}}{1/2} + 5\int \sin 2t\, e^{1/2t}dt = 20e^{1/2t} + 5\int \sin 2t\, e^{1/2t}dt$$

$$\int \sin 2t\, e^{t/2}dt = \sin 2t\,\frac{e^{t/2}}{1/2} - \int 2\cos 2t\,\frac{e^{t/2}}{1/2}dt =$$

$$= 2\sin 2t\, e^{t/2} - 4\cos 2t\,\frac{e^{t/2}}{1/2} + 4\int 2(-\sin 2t)\frac{e^{t/2}}{1/2}dt$$

$$= 2\sin 2t\, e^{t/2} - 8\cos 2t\, e^{t/2} - 16\int \sin 2t\, e^{t/2}dt$$

$$17\int \sin 2t\, e^{t/2}dt = 2\sin 2t\, e^{t/2} - 8\cos 2t\, e^{t/2} \qquad \times 1/17.$$

$$\int (10 + 5\sin 2t)e^{1/2} = 20e^{1/2t} + \frac{10}{17}\sin 2t\, e^{t/2} - \frac{40}{17}\cos 2t\, e^{t/2} + c$$

$$q(t) = \frac{1}{e^{1/2t}}\left[20e^{1/2t} + \frac{10}{17}\sin 2t\, e^{t/2} - \frac{40}{17}\cos 2t\, e^{t/2} + c\right]$$

$$q(t) = 20 + \frac{10}{17}\sin 2t - \frac{40}{17}\cos 2t + ce^{-1/2t}$$

$$q(0) = 0 \qquad 20 - \frac{40}{17} + c = 0 \qquad c = -\frac{300}{17}$$

$$q(t) = 20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t - \frac{300}{17}e^{-t/2}.$$