Diff. Eq. §6 - Existence and Uniqueness of solutions

Q: Is there a solution and is it unique?

Th: Consider the (general) linear DE

(Linear, 7!)

$$y' + p(t)\,y = g(t)$$

If the functions $p(t)$, $g(t)$ are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, there exist a unique solution $y = y(t)$ that also satisfies the initial condition $y(t_0) = y$

Pr: $\mu(t) = \exp \int p(t)\,dt$; $y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s) g(s)\,ds + y_0\right]$. $\boxtimes$

Ex: Find the interval where the initial value problem

$$t y' + 2y = 4t^2 , \quad y(1) = 2$$

has a unique solution.

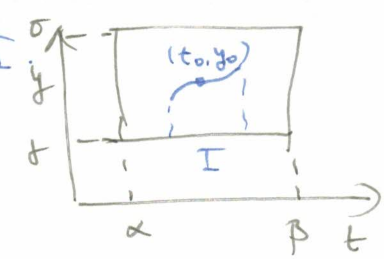
Sol: $y' + \frac{2}{t} y = 4t$, $p(t) = \frac{2}{t}$ (discontinuity at $t=0$), $g(t) = 4t$.

Solution is $y = t^2 + \frac{1}{t^2}$ on $(0, \infty)$. {If the initial condition was $y(-1) = 2$ the same solution formula would be valid on $(-\infty, 0)$

Th: (Nonlinear Eq.) Consider the initial value problem

$$y' = f(t, y) , \quad y(t_0) = y_0$$

If the functions $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous on some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the initial point (t_0, y_0) then there is an interval $I = t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ such that the initial value problem above has a unique solution on I.



rem: Existence (but not uniqueness) follows from the continuity of $f(t,y)$ alone.

Ex: Consider the initial value problem: $y' = y^{1/3}$, $y(0) = 0$. The above theorem does not apply since $f(t,y) = y^{1/3}$, $\partial f/\partial y = \frac{1}{3} y^{-2/3}$ which does not exist at $y=0$. However $f(t,y)$ is continuous so we have $\exists$

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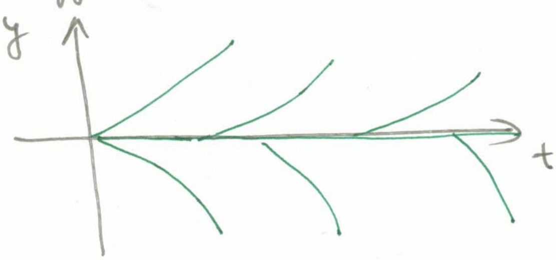
$\frac{dy}{dt} = y^{1/3}$, $\int y^{-1/3} dy = \int dt$, $\frac{3}{2} y^{2/3} = t + C$, $y = \pm\left[\frac{2}{3}(t+c)\right]^{3/2}$

For both $\pm$ the initial condition implies $c = 0$. So we have two solutions $y = \pm\left(\frac{2}{3}t\right)^{3/2}$. But by dividing by $y^{1/3}$ we missed another solution $y = 0$.

In fact, for arbitrary $t_0 > 0$, the functions

$$y(t) = \begin{cases} 0 & 0 \le t \le t_0 \\ \pm\left[\frac{2}{3}(t-t_0)\right]^{3/2} & , t > t_0 \end{cases}$$

are differentiable and solutions of the initial value problem.



(The we have existence, but not uniqueness; infinite family of solutions).

However, if the initial point was not on the line of discontinuity $y = 0$, we would have a unique solution.

Ex: Solve the initial value problem $y' = y^2$, $y(0) = 1$.

The Theorem above guaranties that we have a solution on a t-interval containing t_0.

$\int y^{-2} dy = \int dt$, $-\frac{1}{y} = t + c$; $c = -1$ Thus $y = \frac{1}{1-t}$

So the solution exists on the interval $(-\infty, 1)$; but there is no way to observe this directly from the initial value problem.

In fact if the initial condition is $y(0) = y_0$ the solution is

$y = \frac{y_0}{1 - y_0 t}$ on $(-\infty, 1/y_0)$ if $y_0 > 0$ or on $(1/y_0, \infty)$ if $y_0 < 0$.

The singularities of the solution might depend on the initial conditions.

rem: For a linear equation we have general solution; we have a solution containing one arbitrary constant and all possible

solutions follow by specifying This constant. For nonlinear equati
This may not be the case; even if we have a solution depending
on a constant there might be additional solutions.

rem: For a linear 1st order DE we have an explicit formula for

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s) g(s)\, ds + y_0\right]$$

In the nonlinear case, the best we can hope for is $F(t,y)=0$. This
specifies the solution $y(t)$ implicitly; in general it would be impossibl
to find an analytic solution $y(t)$ in terms of elementary functions.
Finding the interval of validity might also be impossible analytically
→ graphical, → numerical methods.