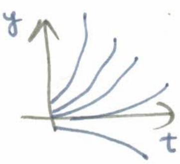
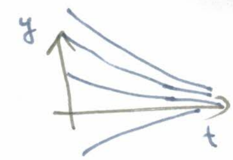
Diff. Eq – §7 – Autonomous Equations

Def: A DE is autonomous if it is of the form
$$\frac{dy}{dt} = f(y)$$

(A) $\frac{dy}{dt} = ry$, $y(0) = y_0$; $\frac{dy}{y} = r\,dt$, $\ln|y| = rt + c$, $|y| = e^c e^{rt}$

$\boxed{y = y_0 e^{rt}}$ – exponential growth or decay

e.g. → population growth; → interest rates
→ radioactive decay

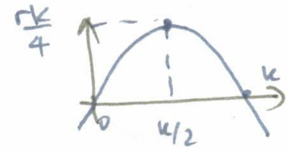
(B) Logistics growth (limitations on resources restrict growth).

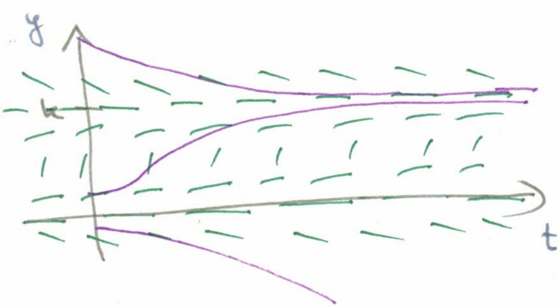
$$\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right)y, \quad r > 0$$

also called critical points.

We have constant solutions $y = k$, $y = 0$. These are equilibrium solutions

To draw the DF, consider $f(y) = r\left(1 - \frac{y}{k}\right)y$,





(In growth modelling k is called carrying capacity; as $k \to \infty$ we recover the the exp growth model.)

Here is an analytic solution: $\int \frac{dy}{(1 - y/k)y} = \int r\,dt$

$$\int \left(\frac{1}{y} + \frac{1/k}{1 - y/k}\right) dy = \int r\,dt; \quad \ln|y| - \ln\left|1 - \frac{y}{k}\right| = rt + c$$

$$\frac{y}{1 - y/k} = c e^{rt}, \quad y(0) = y_0 \quad \frac{y_0}{1 - y_0/k} = c \;\to\; \frac{y}{1 - y/k} = \frac{y_0}{1 - y_0/k} e^{rt}$$

$$\boxed{y = \frac{y_0 k}{y_0 + (k - y_0)e^{-rt}}}$$

The equilibrium solution $y(t) = k$ is asymptotically stable; the solution $y(t) \overset{=0}{} $ is an unstable equilibrium solution.

Diff. Eq. - §7 - Autonomous Equations

Ex: A rumor is started by 2 people and spreads through a college community of 6000 people. The number of people N who have heard the rumor at time t can be modelled by DE:

$$\frac{dN}{dt} = 0.5\left(1 - \frac{N}{6000}\right)N, \quad t \text{ in hours.}$$

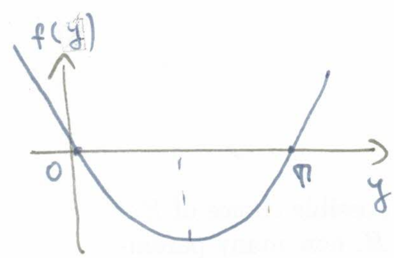
How long it will take for the rumor to reach 80% of the college community?

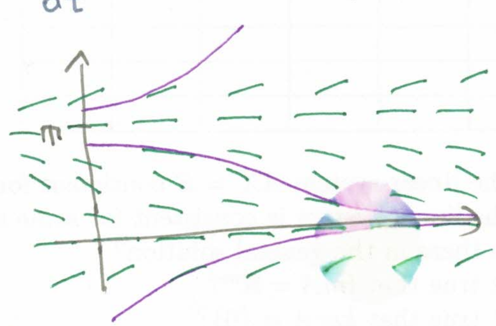
Sol: $y = \dfrac{y_0 k}{y_0 + (k - y_0)e^{-rt}} \rightarrow N(t) = \dfrac{2(6000)}{2 + (6000-2)e^{-0.5t}} = \dfrac{6000}{1 + 2999e^{-0.5t}}$

$4800 = \dfrac{6000}{1 + 2999e^{-0.5t}}, \quad 2999e^{-0.5t} = \dfrac{6000}{4800} - 1 = 0.25$

$-0.5t = \ln\left(\dfrac{0.25}{2999}\right), \quad t = 18.78\,h.$

Ⓒ Critical Threshold. $\dfrac{dy}{dt} = -r\left(1 - \dfrac{y}{T}\right)y, \quad r, T > 0.$





T - is interpreted as a threshold level under which growth does not occur.

The solution can be obtained as before: $y = \dfrac{y_0 T}{y_0 + (T - y_0)e^{rt}}, \quad y(0) = y_0$

Note that when $y_0 > T$ the denominator becomes zero in finite time:

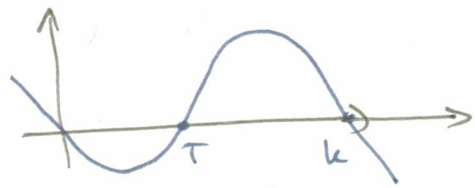
$y_0 + (T - y_0)e^{rt} = 0, \quad t = \dfrac{1}{r}\ln\dfrac{y_0}{y_0 - T}$. Thus the solution has a VA and becomes unbounded in finite time.

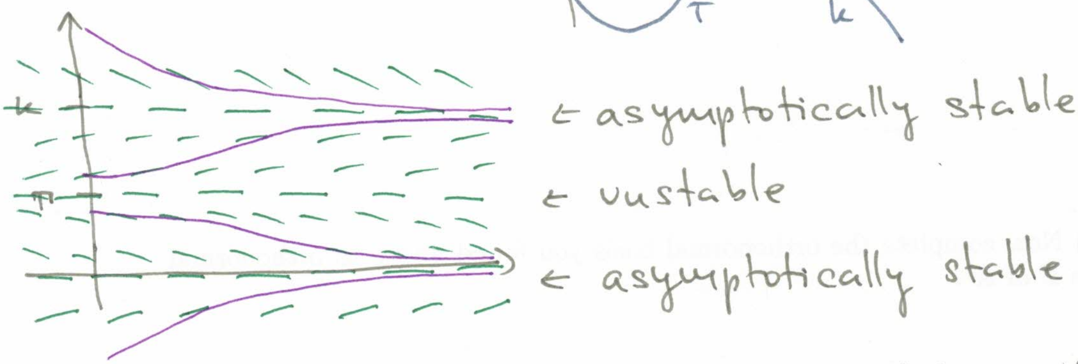
rem: Logistic equations with threshold govern the evolution of a small disturbance in a laminar (smooth) flow. The threshold separates laminar flow from turbulence.

D) Logistical growth with Threshold.

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{k}\right)y \quad , \quad r>0, \quad 0<T<k$$

$$f(y) = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{k}\right)y$$





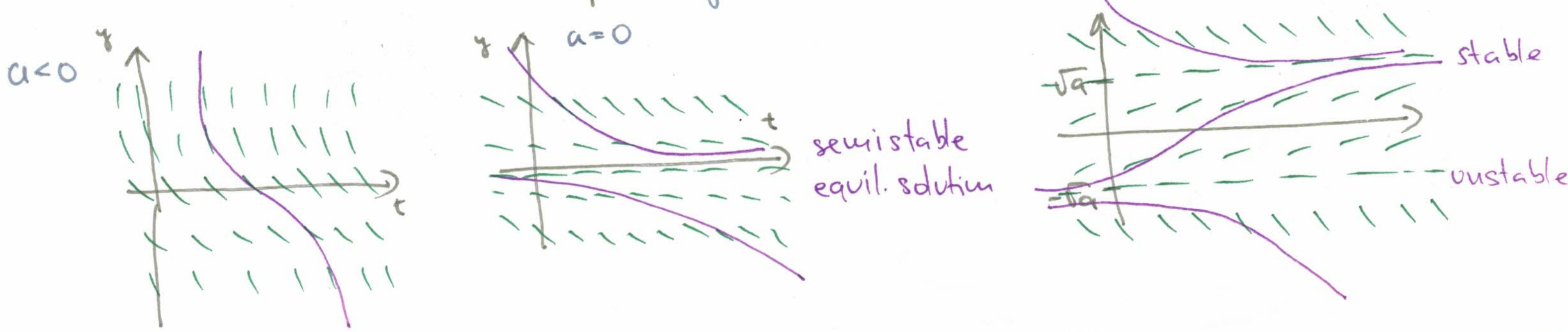
- if a population starts below a Threshold T it goes to extinction
- if the population starts above T it reaches carrying capacity.

Bifurcation points.

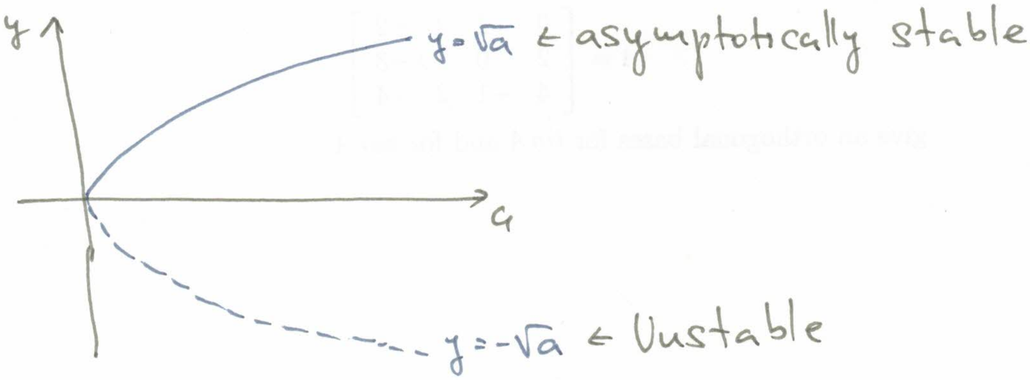
For an equation of the form $y' = f(a,y)$ where a is a parameter, the equilibrium solutions depend on the value of a. At certain values of a, called bifurcation points, critical points come together or separate, and equilibrium solutions may be lost or gained.

- the nature of the solutions of the DE undergoes an abrupt change e.g. a smooth flow may break up and become turbulent.

Ex: Consider $y' = a - y^2$. For $a<0$ there are no critical points; for $a=0$ there is one critical point $y=0$; for $a>0$ we have two critical points $\pm\sqrt{a}$



The **bifurcation diagram** is a plot of the location of the critical points as a function of the parameter a.



The bifurcation at $a = 0$ is called a saddle-node bifurcation.

Note: Bifurcations are discussed in the exercises.