Diff Eq - §8 - Exact equations

Ex: Although the first order equation $y\,dx + x\,dy = 0$ is separable, we can solve it in a different way: let $f(x,y) = xy$. Then

$$df = y\,dx + x\,dy = d(xy)$$

Now if $f = C = \text{const.}$, $df = 0$ i.e. $y\,dx + x\,dy = 0$. Thus the solution is $f = xy = C$. $\quad y = C/x$

Now in general: $z = f(x,y)$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If $z = f(x,y) = C$, $\quad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

and the solution is $f(x,y) = C$.

Ex: $f(x,y) = x^2 - 5xy + y^3 = C$ gives the 1st order DE

$$(2x - 5y)\,dx + (-5x + 3y^2)\,dy = 0 \quad \text{i.e.}$$

$$(2x - 5y) + (-5x + 3y^2)\,y' = 0$$

But how would we recognize that $(2x-5y)dx + (-5x+3y^2)dy$ is the differential of $f(x,y) = x^2 - 5xy + y^3$?

Def. A differential expression $M(x,y)dx + N(x,y)dy$ is an exact differential (in a region R of the xy-plane) if it corresponds to the differential $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ of some function $f(x,y)$ (in R).

A first order DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be an exact equation if the expression on the LHS is an exact differential.

Ex: $x^2y^3dx + x^3y^2dy = 0$ is exact! $f(x,y) = \frac{1}{3}x^3y^3$

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rem: Now for an exact equation $M(x,y)dx + N(x,y)dy = 0$

$\exists f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$.

But for continuous functions we have Clairaut's Theorem:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Th: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous partial derivatives on a rectangle $a<x<b$, $c<y<d$. Then $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad \blacksquare$$

Ex: Solve $2xy\,dx + (x^2-1)\,dy = 0$ (separable, but we will solve it differently)

$M(x,y) = 2xy$, $N(x,y) = x^2-1$ $\rightarrow$ $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ exact!

Thus $\exists f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2-1.$$

Integrate the 1^{st} equation wrt x:

$$f(x,y) = \int 2xy\,dx = x^2y + g(y).$$

Next: $\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$ $\qquad g'(y) = -1$, $g(y) = -y$

Hence: $f(x,y) = x^2y - y$. The solution of the equation is:

$$x^2y - y = c \quad \rightarrow \quad y = \frac{c}{1-x^2}$$

Method of solution: Given an equation $M(x,y) + N(x,y)\,y' = 0$.

① Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

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2. Determine f by integrating $M(x,y)$ wrt x (y = const.)

$$f(x,y) = \int M(x,y)\,dx + g(y).$$

3. Differentiate the result wrt y and compare:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

4. Solve $g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$ for $g(y)$.

5. The solution is $f(x,y) = C$ where

$$f(x,y) = \int M(x,y)\,dx + g(y).$$

Ex: Solve $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$

$$\underbrace{(\cos x \sin x - xy^2)}_{M}\,dx + \underbrace{y(1-x^2)}_{N}\,dy = 0 \qquad \frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial y} = y(1-x^2) \to f(x,y) = \frac{y^2}{2}(1-x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2 \qquad \to h'(x) = \cos x \sin x$$

$$h(x) = \int \cos x \sin x\,dx = -\frac{1}{2}\cos^2 x \to f(x,y) = \frac{y^2(1-x^2)}{2} - \frac{\cos^2 x}{2}$$

$$\frac{y^2(1-x^2)}{2} - \frac{\cos^2 x}{2} = C, \quad y(0) = 2 \to \frac{4}{2} - \frac{1}{2} = C \qquad C = 3/2$$

$$y^2(1-x^2) - \cos^2 x = 3, \quad y = \sqrt{\frac{3+\cos^2 x}{1-x^2}}$$

Integrating factors. Say $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact.
Multiply by an integrating factor: $\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$.
We have exactness when: $(\mu M)_y = (\mu N)_x$ i.e.

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N \quad \text{or} \quad \mu_x N - \mu_y M = (M_y - N_x)\mu.$$

This now is a partial differential equation for μ.

Let's make the simplifying assumption μ does not depend on y, i.e. $\mu_y = 0$. Then

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \quad \leftarrow \text{This now is linear, ordinary DE.}$$

ex: $xy\,dx + (2x^2+3y^2-20)\,dy = 0$ $\quad \frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = 4x$ Not exact.

Here the integrating factor is a function of y:

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu = \frac{4x - x}{xy}\mu = \frac{3}{y}\mu$$

$\mu(y) = e^{\int \frac{3}{y}dy} = e^{\ln y^3} = y^3$. After multiplying we get:

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 4xy^3 = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x} = xy^4 \rightarrow f(x,y) = \frac{x^2y^4}{2} + g(y)$$

$$\frac{\partial f}{\partial y} = 2x^2y^3 + g'(y) = 2x^2y^3 + 3y^5 - 20y^3$$

$$g'(y) = 3y^5 - 20y^3 \qquad g(y) = \frac{y^6}{2} - 5y^4$$

$$f(x,y) = \boxed{\frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 = C}$$