Diff. Eq. – §9 – Euler's Method

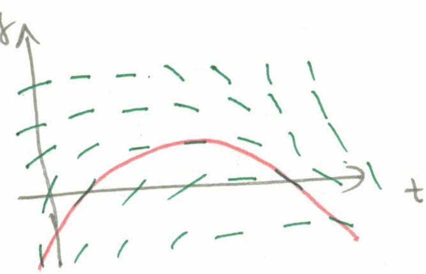
$\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$

→ we have hope for analytical solution if this equation is (or can transformed to) linear, separable, exact, ... ; the vast majority of 1^{st} order initial value problems cannot be solved by analytic means.

→ we have however existence/uniqueness theorem for the solutions under continuity of $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$, (on some interval containi (t_0, y_0)).

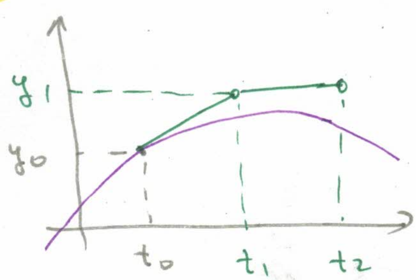
Ex: + The DE: $\frac{dy}{dt} = 3 - 2t - 0.5y$

The direction field of this DE is:



Idea: If we link the tangent segments we will obtain an approximation for the solution. But how far does the approximation deviate from the true solution?

constr: Euler's method:



$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$

If t_1 is closed enough to t_0 the approximation will be reasonably precise. Now repeat:

$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$

$\vdots$

$\boxed{y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)}$ $\quad n = 0, 1, 2 \ldots$

Typically the step size is uniform $t_{n+1} = t_n + h$ $\forall n$:

$\boxed{y_{n+1} = y_n + f_n h, \quad f_n = f(t_n, y_n), \quad n = 0, 1, 2 \ldots}$ Euler's formula.

Ex: + The initial value problem $y' = 3 - 2t - y/2$, $y(0) = 1$.

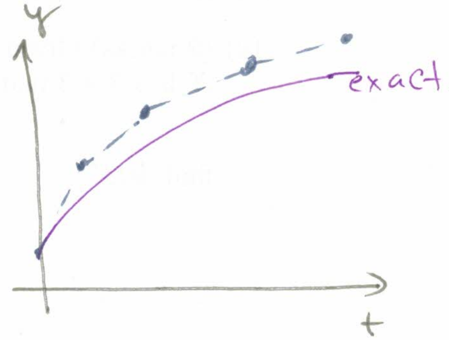
This is linear, we can solve it: $\mu(t) = e^{t/2}$

$y(t) = 14 - 4t - 13e^{-t/2}$ is the solution.

Diff Eq - §9 - Euler's Method

Let us use Eulers method with $h = 0.2$.

n	t_n	y_n	f_n " y'_n	exact
0	0	1	2.5	1
1	0.2	1.5	1.85	1.437
2	0.4	1.87	1.265	1.757
3	0.6	2.123	0.7385	1.969
4	0.8	2.2707	0.2647	2.086
5	1	2.3236		2.1151



Ex. cont'd: $y' = 3 - 2t - 0.5y$, $y(0) = 1$ $\rightarrow$ $y(t) = 14 - \cancel{3}t_{4t} - 13e^{-t/2}$

t	$h=0.1$	$h=0.05$	$h=0.01$	exact
0	1	1	1	1
1	2.216	2.165	2.125	2.115
2	1.340	1.278	1.230	1.218
3	-0.790	-0.846	-0.890	-0.901
4	-3.671	-3.715	-3.751	-3.760
5	-7.000	-7.034	-7.060	-7.067

$\rightarrow$ decreasing step-size increases the accuracy of the approximation

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Ex: $y' = 4 - t + 2y$, $y(0) = 1$ $\rightarrow$ $y(t) = -7/4 + 1/2\, t + 11/4\, e^{2t}$

t	$h=0.1$	$h=0.05$	$h=0.01$	exact
0	1	1	1	1
1	15.777	17.251	18.673	19.070
2	104.678	123.713	143.584	149.395
3	652.535	837.075	1045.395	1107.179
4	4042.122	5633.351	7575.577	8197.884
5	25026.95	37897.43	54881.32	60573.53

why: observe that $y(t) = 14 - 3t - ce^{-t/2}$ is a converging family as $t \to \infty$; in our given case $c = -13$, but that doesn't matter much since the solutions are converging in $14 - 3t$ regardless of c.

On the other hand $y = -7/4 + t/2 + ce^{2t}$ is a divergent family when $t \to \infty$ (in our case $c = 11/4$), but the approximations in the Euler method moved us away). Ch8 has analysis of errors.

There two sources of error in Euler algorithm

(1) The formula used in the algorithm is a (straight-line) approximation to the actual solution.

(2) Except for the first step, the input data for the calculation uses approximation for the actual values of the solution at the specified points.

Def: Let $y=\phi(t)$ be the exact solution and let y_n be the approximate value at time t_n. The global truncation error is

$$E_n = \phi(t_n) - y_n.$$

Assuming that the input data for the n'th step is accurate, i.e. $y_{n-1} = \phi(t_{n-1})$, the local truncation error is

$$e_n = \phi(t_n) - y_n$$

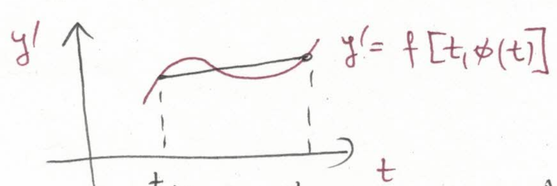
For the Euler method $e_n \sim h^2$, $E_n \sim h$ (first order method).

$y' = f(t,y)$
$y(t_0) = y_0$

Now in the Euler method we approximate

$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f[t, \phi(t)]\, dt$$

with $y_{n+1} = y_n + h f(t_n, y_n)$



so we are using the approximate value $f(t_n, y_n)$ at the left end point. A better approximation for the area under $y' = f(t, \phi(t))$ is obtained by:

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} h$$

But we don't know y_{n+1}. So

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_n + h, y_n + h f(t_n, y_n))}{2} h$$

This improved Euler method has $e_n \sim h^3$ and $E_n \sim h^2$ (so this is a second order method).

The Runge-Kutta formula uses a weigted average which can be interpreted as average slope:

$$y_{n+1} = y_n + h\left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\right),$$

$$k_{n1} = f(t_n, y_n)$$

$$k_{n2} = f\left(t_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}hk_{n1}\right)$$

$$k_{n3} = f\left(t_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}hk_{n2}\right)$$

$$k_{n4} = f(t_n + h,\ y_n + hk_{n3}).$$

Here $e_n \sim h^5$ and $E_n \sim h^4$ so this is a 4th order method.

rem: Adaptive RK methods modify the step size automatically as the computation proceeds, so to maintain a local truncation error below a specified tolerance level. This requires the estimation of the local truncation error at each step. One way to do this is to use a 5th order method for the step forward and the difference between the two results is an estimate of the error.

The 4th, 5th order RK pair is called RKF (Runge-Kutta-Fehlberg) method.

rem: ode 45 in Matlab is a particular implementation based on the Dormand - Prince pair.

Improvements on the Euler method.

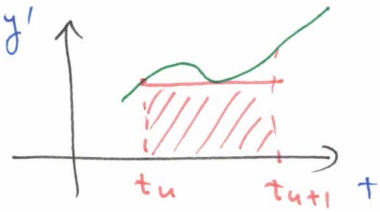
Initial value problem: $y' = f(t,y)$, $y(t_0) = y_0$.

Let $Y(t)$ be the exact solution. Then

$$Y(t_{n+1}) = Y(t_n) + \int_{t_n}^{t_{n+1}} f[t, Y(t)]\, dt$$

But we approximate this with the Euler formula

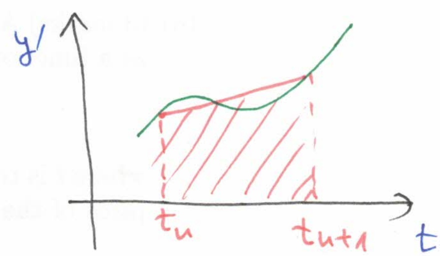
$$y_{n+1} = y_n + h f(t_n, y_n)$$



so we approximate the area under the curve by the area of the rectangle with height value of the function at left endpoint.

Better approximation would be

$$y_{n+1} = y_n + h\,\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$$



But we don't know y_{n+1} on the RHS.

So we will approximate using Euler's formula: $y_{n+1} = y_n + hf_n$

$$y_{n+1} = y_n + h\,\frac{f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))}{2}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f_n + f(t_n + h, y_n + hf_n)\right).$$

Euler's method is a first order method: $E_n = Y(t_n) - y_n$ - global truncation error.

$$|E_n| \le Kh.$$

Improved Euler's method is a second order method:

$$|E_n| \le K_1 h^2.$$