DIFFERENTIAL EQUATIONS, H22, TEST 1

(1) (2.5 marks) Find the solution $y(t)$ of the initial value problem

$$y' + \frac{y}{t} = \frac{2}{4-t^2}, \quad y(1) = \ln(2).$$

On what interval is the solution you found defined?

(1/2) $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln|t| + C} = Ct \rightarrow \mu(t) = t$

(1/2) $\frac{d}{dt}[ty] = \frac{2t}{4-t^2}$

(1/2) $ty = \int \frac{2t}{4-t^2} dt = -\ln|4-t^2| + C$

$1 \cdot \ln 2 = -\ln 3 + C \Rightarrow C = \ln 6$

(1/2) $y(t) = \frac{1}{t}\left[-\ln(4-t^2) + \ln 6\right] = \frac{1}{t} \ln \frac{6}{4-t^2}$

(1/2) The interval of validity of this solution is $(0,2)$.

(2) (2 marks) One model for tumor growth is the Gompertz equation

$$\frac{dR}{dt} = -aR\ln(R/k),$$

where $R = R(t)$ is the tumor radius, and a and k are positive constants.

a) Solve the Gompertz equation for $R(t)$.

b) Determine the limit at $t \to \infty$ of your solution.

$\frac{1}{2}$ $\qquad \dfrac{dR}{R\ln\frac{R}{k}} = -a\,dt \qquad$ $\frac{1}{2}$ $\qquad u = \ln\dfrac{R}{k} \qquad du = \dfrac{1}{R}\,dR$

$\dfrac{du}{u} = -a\,dt \qquad \ln|u| = -at + C$

$|u| = e^{c-at}, \qquad \ln\dfrac{R}{k} = ce^{-at}$

$\frac{1}{2}$ $\qquad R(t) = k\,e^{ce^{-at}}$

$\frac{1}{2}$ $\qquad \lim\limits_{t\to\infty} R(t) = k$

(3) (2.5 marks) Find a solution for the initial value problem

$$xy^2 + 3x^2y + (x+y)x^2y' = 0, \quad y(1) = -1.$$

What is the interval of existence of the solution?

$$\underbrace{(xy^2 + 3x^2y)}_{M}\,dx + \underbrace{(x^3 + x^2y)}_{N}\,dy = 0$$

(1/2) $$\frac{\partial M}{\partial y} = 2xy + 3x^2 = \frac{\partial N}{\partial x} \qquad \text{exact!}$$

(1/2) $$f(x,y) = \int (xy^2 + 3x^2y)\,dx = \frac{x^2y^2}{2} + x^3y + g(y)$$

$$\frac{\partial f}{\partial y} = x^2y + x^3 + g'(y) = x^3 + x^2y \quad \Rightarrow g(y) = C$$

(1/2) $$f(x,y) = \frac{x^2y^2}{2} + x^3y = C$$

$$y(1) = -1 \Rightarrow \frac{1}{2} - 1 = C \qquad C = -\tfrac{1}{2}$$

(1/2) $$\frac{x^2y^2}{2} + x^3y = -\frac{1}{2} \qquad y^2 + 2xy + \frac{1}{x^2} = 0$$

$$y_{1,2} = \frac{-2x \pm \sqrt{4x^2 - 4/x^2}}{2} = -x \pm \sqrt{x^2 - \frac{1}{x^2}}$$

$$x^2 - \frac{1}{x^2} \geq 0 \qquad x^4 \geq 1 \qquad x \geq 1$$

(1/2) The interval of validity is $[1, \infty)$ for either solution.

(4) (2.5 marks) Consider the differential equation

$$\frac{dx}{dt} = x(1-x) - h,$$

which describes logistic population growth with harvesting. The existence of equilibrium solutions depends on the harvesting parameter h.

a) Determine the value of h which corresponds to a bifurcation point.

b) For values of h below and above the bifurcation point plot the directional field of the equation together with several integral curves.

c) Draw the bifurcation diagram of this DE, i.e. plot the location of the critical points versus the parameter h.

1/2

$$x^2 - x + h = 0, \qquad x_{1,2} = \frac{1 \pm \sqrt{1-4h}}{2}$$

For $h < 1/4$ there are two equilibrium solutions.

For $h = 1/4$ there is one equilibrium solution at $x = 1/2$.

For $h > 1/4$ there are no equilibrium solutions.

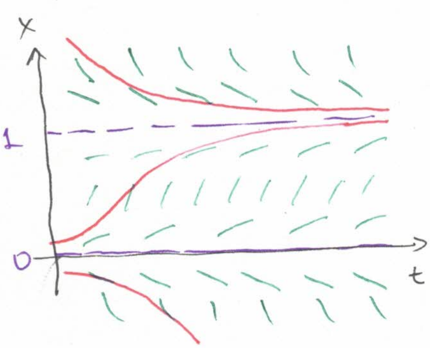
1/2

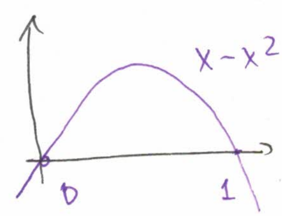
$h = 0$

$x_1 = 0$

$x_2 = 1$

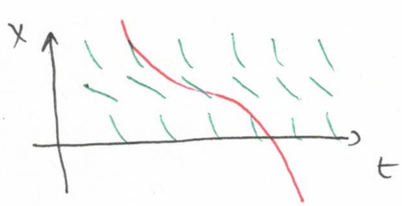
1/2

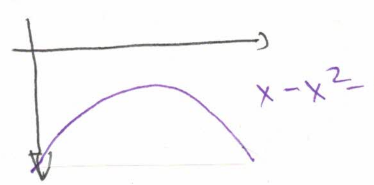




$h = 1$

1/2





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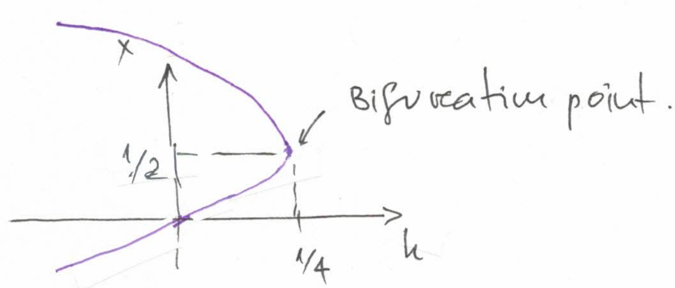


(5) (2.5 marks) Determine the solution of the initial value problem

$$2y'' + y' + 3y = 0, \quad y(0) = -3, \; y'(0) = 1$$

and plot a sketch of the solution.

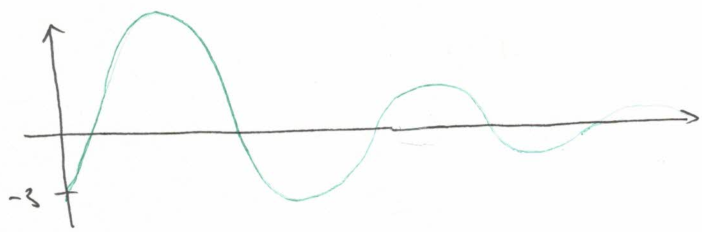
1/2 $2r^2 + r + 3 = 0 \qquad r_{1,2} = \dfrac{-1 \pm \sqrt{1-24}}{4} = -\dfrac{1}{4} \pm \dfrac{\sqrt{23}}{4} i$

1/2 $y(t) = c_1 e^{-t/4} \cos \dfrac{\sqrt{23}}{4} t + c_2 e^{-t/4} \sin \dfrac{\sqrt{23}}{4} t$

$-3 = y(0) = c_1 \; , \; c_1 = -3$

1/2 $1 = y'(0) = -\dfrac{c_1}{4} + c_2 \cdot \dfrac{\sqrt{23}}{4} \; , \; c_2 = \dfrac{1}{\sqrt{23}}$

1/2 $y(t) = -3 e^{-t/4} \cos \dfrac{\sqrt{23}}{4} t + \dfrac{1}{\sqrt{23}} e^{-t/4} \sin \dfrac{\sqrt{23}}{4} t$

1/2 

(6) (2 marks) Determine the general solution of

$$y'' + ty' + y = 0,$$

given that

$$y_1(t) = e^{-\frac{t^2}{2}}$$

is one solution.

$y_2(t) = v(t)\, y_1(t)$

1/2 $y_2' = v' y_1 + v y_1'$; $y_2'' = v'' y_1 + 2v' y_1' + v y_1''$

$v'' y_1 + 2v' y_1' + v y_1'' + t v' y_1 + t v y_1' + v y_1 =$

$= v'' y_1 + 2 v' y_1' + t v' y_1 + v \underbrace{\cancel{(y_1'' + t y_1' + y_1)}}_{\to 0} = 0$

1/2 $v'' y_1 + v'(2 y_1' + t y_1) = 0$ $\quad w = v'$

$w' e^{-t^2/2} + w\,(2\cdot(-t) + t)\, e^{-t^2/2} = 0$

$w' - wt = 0 \qquad \frac{dw}{w} = t\,dt \qquad \ln|w| = \frac{t^2}{2} + c$

1/2 $w(t) = e^{t^2/2}$

$v(t) = \int e^{t^2/2}\,dt \qquad y_2(t) = e^{-t^2/2} \int e^{t^2/2}\,dt$

1/2 $y(t) = c_1 e^{-t^2/2} + c_2 e^{-t^2/2} \int e^{t^2/2}\,dt$

(7) (3 marks) Solve the initial value problem

$$y'' - 4y = 5e^{-2t}, \quad y(0) = 1,\ y'(0) = 1.$$

Compute the limit at $t \to \infty$ of the solution. Sketch the graph of the solution.

1/2 $r^2 = 4 \qquad r_{1,2} = \pm 2 \qquad y_{hom}(t) = c_1 e^{2t} + c_2 e^{-2t}$

1/2 Ausatz: $Y(t) = cte^{-2t} \qquad Y'(t) = ce^{-2t} - 2cte^{-2t}$

$$Y''(t) = -4ce^{-2t} + 4cte^{-2t}$$

1/2 $-4ce^{-2t} + \cancel{4cte^{-2t}} - \cancel{4cte^{-2t}} = 5e^{-2t} \qquad c = -5/4$

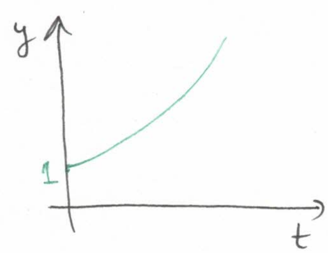
$y(t) = c_1 e^{2t} + c_2 e^{-2t} - \frac{5}{4} te^{-2t}$

$1 = y(0) = c_1 + c_2 ; \qquad 1 = y'(0) = 2c_1 - 2c_2 - 5/4$

1/2 $c_1 = \frac{17}{16} \qquad c_2 = -\frac{1}{16}$

1/2 $y(t) = \frac{17}{16} e^{2t} - \frac{1}{16} e^{-2t} - \frac{5}{4} te^{-2t}$

1/2



(8) (3 marks) Solve the initial value problem

$$y'' - 2y' + y = \frac{e^t}{1+t^2}, \quad y(0) = 2,\ y'(0) = 1$$

$r^2 - 2r + 1 = 0 \qquad (r-1)^2 = 0 \qquad r_{1,2} = 1$

(1/2) $y_1(t) = e^t \qquad y_2(t) = te^t$

(1/2) $W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$

(1/2) $u_1(t) = -\int \frac{te^t \cdot e^t/1+t^2}{e^{2t}}\,dt = -\int \frac{t}{1+t^2}\,dt = -\frac{1}{2}\ln(1+t^2) + C$

(1/2) $u_2(t) = \int \frac{e^t \cdot e^t/1+t^2}{e^{2t}}\,dt = \int \frac{1}{1+t^2}\,dt = \arctan t + C$

$Y(t) = \left(-\frac{1}{2}\ln(1+t^2)\right)e^t + t\arctan t\, e^t$

(1/2) $y(t) = C_1 e^t + C_2 te^t - \frac{1}{2}\ln(1+t^2)e^t + t\arctan(t) \cdot e^t$

$2 = y(0) = C_1$

$1 = C_1 + C_2 \qquad C_2 = -1$

(1/2) $y(t) = 2e^t - te^t - \frac{1}{2}\ln(1+t^2)e^t + te^t \arctan t.$