DIFFERENTIAL EQUATIONS, H22, TEST 2

Name: ________________________ Student number______________________

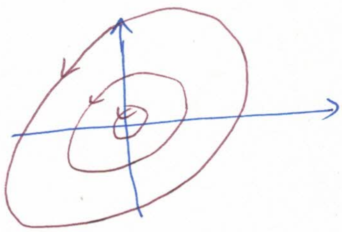
(1) (3.5 marks) The coefficient matrix of the following system of differential equations depends on a parameter α.
a) Determine the eigenvalues in terms of α.
b) Find the critical values of α where the qualitative nature of the phase portrait for the system changes.
c) Draw qualitative phase portraits for this system for values of α taken in the intervals outside of the critical points.

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}.$$

$$\det \begin{pmatrix} 2-\lambda & -5 \\ \alpha & -2-\lambda \end{pmatrix} = \lambda^2 - 4 + 5\alpha \qquad \lambda_{1,2} = \pm\sqrt{-5\alpha + 4}$$

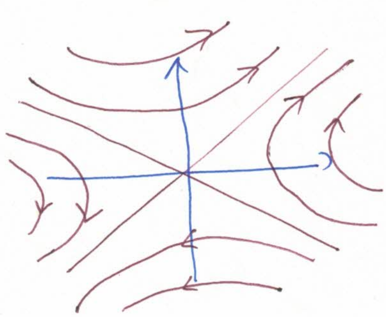
$\alpha_{cr} = 4/5$

Case 1: $\alpha > 4/5$ $\quad \lambda_{1,2} = \pm\sqrt{5\alpha - 4}\, i$



Center.

Case 2: $\alpha < 4/5$ $\quad \lambda_{1,2} = \pm\sqrt{-5\alpha + 4}$



saddle point.

(2) (4.5 marks) The motion of a mechanical system driven by a periodic external force can be modelled as a initial value problem

$$u'' + \omega_0^2 u = F\cos\omega t, \quad u(0) = 0, \; u'(0) = 0,$$

where F, ω_0 and ω are constants and $\omega \neq \omega_0$.
a) Solve the initial value problem. The result should be a function $u(t)$ with the constants F, ω_0, ω as parameters in the function.
b) Now let $F = 1$, $\omega_0 = 1$ and $\omega = 1.1$. Substitute these values in your solution $u(t)$. Sketch the graph of $u(t)$ clearly indicating the maximum value(s) on the graph and the period if $u(t)$ is periodic.

Homogeneous equation: $u'' + \omega_0^2 u = 0$

General solution of the homogeneous equation

$u_h(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$

Particular solution; Ansatz: $Y(t) = A\cos\omega t + B\sin\omega t$

$Y'(t) = -A\omega\sin\omega t + B\omega\cos\omega t, \quad Y''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

$\cos\omega t: \; -A\omega^2 + A\omega_0^2 = F \qquad A = \dfrac{F}{\omega_0^2 - \omega^2}$

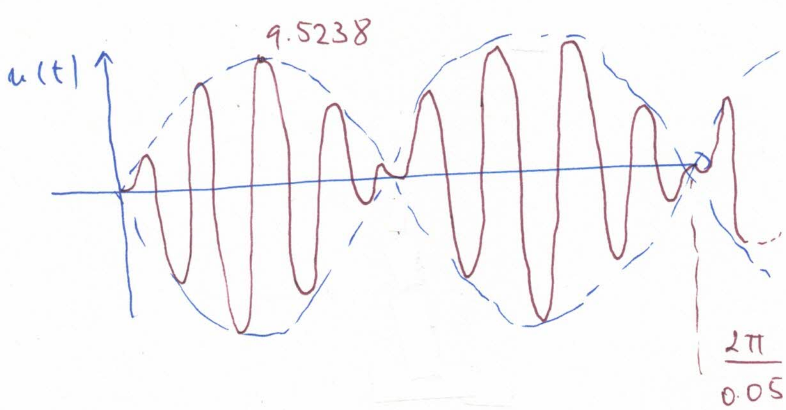
$\sin\omega t: \; -B\omega^2 + B\omega_0^2 = 0 \qquad B = 0$

$u(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \dfrac{F}{\omega_0^2 - \omega^2}\cos\omega t$

$u(0) = c_1 + \dfrac{F}{\omega_0^2 - \omega^2} = 0 \qquad c_1 = -F/(\omega_0^2 - \omega^2)\,; \qquad u'(0) = c_2\omega_0 = 0 \quad c_2 = 0$

$u(t) = \dfrac{F}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t) = \dfrac{2F_0}{\omega_0^2 - \omega^2}\sin\dfrac{(\omega_0 - \omega)t}{2}\sin\dfrac{(\omega_0 + \omega)t}{2}$

$u(t) = 9.5238\sin(0.05t)\sin(1.05t)$



$\dfrac{2\pi}{0.05} = 40\pi$

(3) (4 marks) Solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

Draw the phase portrait of this linear system of differential equations emphasizing the particular trajectory selected by the initial conditions.

$$\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 2 \qquad \lambda_{1,2} = \frac{2 \pm \sqrt{4-4\cdot 2}}{2} = 1 \pm i$$

$$\lambda_1 = 1+i \quad \left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

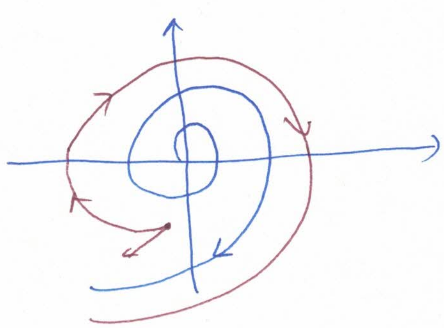
$$\vec{x}^{(1)}(t) = e^{(1+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^t(\cos t + i \sin t)\begin{pmatrix} 1 \\ i \end{pmatrix} =$$

$$= e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \qquad c_1 = -1, \ c_2 = -2$$

$$\vec{x}(t) = -e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - 2e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \qquad \vec{x}'(0) = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$



(4) (4 marks) a) Find the solution of the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -2.5 & 1.5 \\ -1.5 & 0.5 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

b) Draw a phase portrait of this system of DE's emphasizing the trajectory selected by the initial conditions.

$$\det\begin{pmatrix} -2.5-\lambda & 1.5 \\ -1.5 & 0.5-\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 1 = 0 \qquad \lambda_1 = \lambda_2 = -1$$

$$\lambda = -1 \quad \left(\begin{array}{cc|c} -1.5 & 1.5 & 0 \\ -1.5 & 1.5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x_2 \text{ - missing}$$

Solving for a generalized eigenvector.

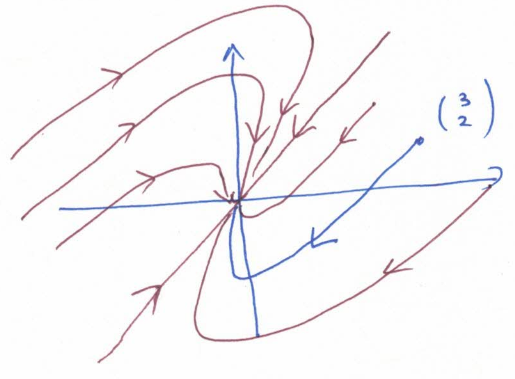
$$\left(\begin{array}{cc|c} -1.5 & 1.5 & 1 \\ -1.5 & 1.5 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -2/3 \\ 0 & 0 & 0 \end{array}\right) \qquad \eta = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix}$$

$$\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left(t e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -2/3 \\ 0 \end{pmatrix}\right)$$

$$\vec{x}(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -2/3 \\ 0 \end{pmatrix} \Rightarrow c_1 = 2, \quad c_2 = -3/2$$

$$\vec{x}(t) = 2e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{2}\left(t e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -2/3 \\ 0 \end{pmatrix}\right)$$

$$\vec{x}(t) = e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} - \frac{3}{2} t e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



(5) (4 marks) Determine the matrix exponential e^{At} for the matrix A:

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}.$$

Next use this result to solve the initial value problem

$$\begin{aligned} x_1' &= 5x_1 - x_2 \\ x_2' &= 3x_1 + x_2, \end{aligned} \qquad x_1(0) = -2,\ x_2(0) = 4.$$

$$\det\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4) \qquad \lambda_1 = 2,\ \lambda_2 = 4$$

$$\lambda_1 = 2 \quad \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\lambda_2 = 4 \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$e^{At} = P e^{Dt} P^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$e^{At} = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} & e^{2t} - e^{4t} \\ -3e^{2t} + 3e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}$$

$$\vec{x}(t) = e^{At}\vec{x}(0) = e^{At}\begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} 3e^{2t} - 5e^{4t} \\ 9e^{2t} - 5e^{4t} \end{pmatrix}$$

(6) (4 marks) Determine the solution of the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

$$\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \qquad \lambda_1 = 1,\ \lambda_2 = -1$$

$$\lambda_1 = 1 \quad \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad x_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \qquad \Psi^{-1}(t) = \frac{1}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix}$$

$$\vec{x}(t) = \Psi(t)\Psi^{-1}(0)\vec{x}(0) + \Psi(t)\int_0^t \Psi^{-1}(s)\vec{g}(s)\,ds =$$

$$= \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}\int_0^t \frac{1}{2}\begin{pmatrix} e^{-s} & -e^{-s} \\ -e^s & 3e^s \end{pmatrix}\begin{pmatrix} e^{2s} \\ 1 \end{pmatrix} ds$$

$$= \frac{1}{2}\begin{pmatrix} -3e^t + e^{-t} \\ -e^t + e^{-t} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}\begin{pmatrix} e^t + e^{-t} - 2 \\ -\frac{1}{3}e^{3t} + 3e^t - \frac{8}{3} \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{pmatrix}$$

(7) (4 marks) Use Laplace transform to solve the initial value problem

$$y'' - 2y' + 5y = -8e^{-t}, \qquad y(0) = 2,\ y'(0) = 12.$$

$$\mathcal{L}\{y'' - 2y' + 5y\} = \mathcal{L}\{-8e^{-t}\}$$

$$(s^2Y(s) - 2s - 12) - 2(sY(s) - 2) + 5Y(s) = \frac{-8}{s+1}$$

$$Y(s) = \frac{2s^2 + 10s}{(s^2-2s+5)(s+1)} \qquad \frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s+1}$$

s^2: $A + C = 2$, s: $-5C + 2 - C - 2C = 10$, 1: $B + 5C = 0$

$\Rightarrow C = -1$, $B = 5$, $A = 3$

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{3s+5}{s^2-2s+5} - \frac{1}{s+1} = \frac{3(s-1)+8}{(s-1)^2+2^2} - \frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{8}{(s-1)^2+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$y(t) = 3e^t\cos 2t + 4e^t\sin 2t - e^{-t}$$

(8) (4.5 marks) The current I in an LC series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t), \qquad I(0) = 0,\ I'(0) = 0,$$

where

$$g(t) = \begin{cases} 1 & 0 \le t < 1, \\ -1 & 1 \le t < 2, \\ 0 & 2 \le t. \end{cases}$$

a) Determine the current as a function of time t.

b) Is the current a continuous function of time? Check $I(t)$ for continuity at the points $t = 1$ and $t = 2$ where the nonhomogeneous term is discontinuous.

$g(t) = 1 - 2u_1(t) + u_2(t) \qquad \mathcal{L}\{I(t)\} = J(s)$

$\mathcal{L}\{I''(t) + 4I(t)\} = \mathcal{L}\{g(t)\}$

$(s^2+4)J(s) = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$

$J(s) = \frac{1}{s(s^2+4)} - 2\frac{e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)}$

Let $F(s) = \frac{1}{s(s^2+4)} = \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$; $f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{4} - \frac{1}{4}\cos 2t$

$I(t) = \mathcal{L}^{-1}\{F(s) - 2e^{-s}F(s) + e^{-2s}F(s)\} =$

$= f(t) - 2u_1(t)f(t-1) + u_2(t)f(t-2)$

$I(t) = \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left(\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right)u_1(t) + \left(\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right)u_2(t)$

$\lim_{t\to 1^-} I(t) = \frac{1}{4} - \frac{1}{4}\cos 2 = \lim_{t\to 1^+} I(t)$

$\lim_{t\to 2^-} I(t) = -\frac{1}{4} + \frac{1}{4}\cos(4) + \frac{1}{2}\cos(2) = \lim_{t\to 2^+} I(t)$

$I(t)$ is a continuous function of time.

(9) (3.5 marks) Use the Laplace transform to solve the initial value problem

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0, \ y'(0) = 1$$

$$\mathcal{L}\{y'' + 2y'\} = \mathcal{L}\{\delta(t-1)\}$$

$$s^2 Y(s) - 1 + 2s Y(s) = e^{-s}$$

$$(s^2 + 2s) Y(s) = e^{-s} + 1 \qquad Y(s) = \frac{1}{s^2+2s} + \frac{e^{-s}}{s^2+2s}$$

$$\frac{1}{s^2+2s} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+2}$$

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+2}\right\}$$

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} + \frac{1}{2}u_1(t) - \frac{1}{2}u_1(t)\,e^{-2(t-1)}$$