DiscMath - HW6 - Clex 10 - Solutions

① We know that $\mathbb{Z}$ is countable. Let $\{u_1, u_2, \ldots\}$ be an enumeration of $\mathbb{Z}$. Arrange the elements of $\mathbb{Z}^2$ in a corner

$$\begin{array}{llllll}
(u_1,u_1) \to (u_1,u_2) & (u_1,u_3) \to (u_1,u_4) & \ldots \\
(u_2,u_1) & (u_2,u_2) & (u_2,u_3) & (u_2,u_4) & \ldots \\
\downarrow & \nearrow & \swarrow & \searrow & \downarrow \\
(u_3,u_1) & (u_3,u_2) & (u_3,u_3) & (u_3,u_4) & \ldots \\
\swarrow & \nearrow & \swarrow & \searrow & \\
(u_4,u_1) & (u_4,u_2) & (u_4,u_3) & (u_4,u_4) & \ldots \\
\downarrow & \nearrow & & & \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

Following the arrows gives an enumeration of $\mathbb{Z}^2$.

② Consider the following mapping from $P(S)$ to 2^S: for a subset $A \in P(S)$ the function $f_A : S \to 2 = \{0,1\}$ is

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$A \to f_A$ is clearly a bijection from $P(S)$ to 2^S.

③ The new bijection is

$$h(1) = z\,; \qquad h(n) = g(n-1),\ n \geq 2$$

④ Let the Hilbert Hotel with countably infinitely many rooms be f and let countably infinitely many busses each with countably infinitel many new guests arrive. Move the preexisting guests to rooms 1, 3, 6, 10, 21, 28, ... creating "gaps" of empty rooms of lengths 1, 2, 3, 4, 5, .. Accommodate the passengers from the 1st bus by putting them in the first available roo in a gap (one per gap) starting with the 1st gap; accommodate the passenge from the second bus by putting them in the 1st available room in each gap (starting with the 2nd gap), etc. The first dozen rooms have the following gues

All guests (old and new) will be accomodated establishing a bijection between a countably infinite union of countably infinite sets and a countable set