DISCRETE MATHEMATICS, CLASS EXERCISE 10

(1) Prove that the set $\mathbb{Z}^2$ is countably infinite.

(2) Let $\mathcal{P}(S)$ be the set of all substes of a set S, and let T be the set of all functions from S to the set $\{0, 1\}$. Prove that $\mathcal{P}(S)$ and T have the same cardinality.

(3) Imagine that there is a hotel, called *Hilbert Hotel*, that contains a countably infinite number of rooms. The nice featute of this hotel is that even when it is full, it can always take another guest. For convenience suppose that the rooms are numbered $1, 2, 3, \ldots$ and G is the set of guests. If the hotel is full then we have a bijection $g : \mathbb{N} \to G$. Suppose that a new guest z arrives. Find a new bijection $h : \mathbb{N} \to G \cup \{z\}$ to accomodate the new guest.

(4) (Bonus) Using a Hilbert Hotel type of argument prove that a countably infinite union of countable sets is countable.