Discrete Math - Clex 13 - Solutions

(1) False. Consider the set $A = \{a,b\}$ and the relations $R = \{(a,b)\}$, $S = \{(b,a)\}$. Then both R and S are trivially transitive, but $R \cup S = \{(a,b),(b,a)\}$ is not.

(2) Reflexive $m=n$, $m^2-n^2=0$, $3 | 0$ ✓

Symmetric $3 | m^2-n^2 = -(n^2-m^2) \Rightarrow 3 | n^2-m^2$ ✓

Transitive $3|(m^2-n^2)$, $3|(n^2-l^2) \Rightarrow 3|(m^2-l^2) = (m^2-n^2)+(n^2-l^2)$

Now $3|(m-n) \Rightarrow m\,D\,n$ and so the equivalence classes of D are coarser (larger) than the equivalence classes of mod 3.

Notice that $3|2^2-1^2 \Rightarrow [1]\,D\,[2]$. The equivalence classes of D are $[0]$, $[1]$.

(3) $x-x = 0 \in \mathbb{Q} \Rightarrow$ The relation is reflexive.

$x-y \in \mathbb{Q} \Rightarrow (y-x) = -(x-y) \in \mathbb{Q} \Rightarrow$ The relation is symmetric.

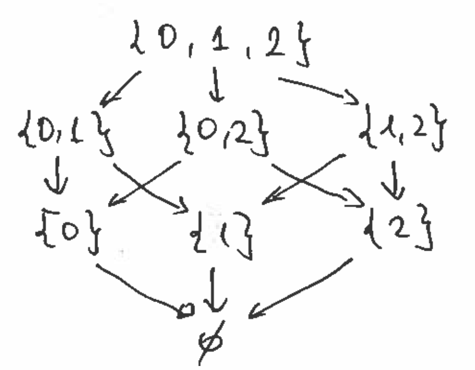
$x-y = \frac{p}{q} \in \mathbb{Q}$, $y-z = \frac{r}{s} \in \mathbb{Q} \Rightarrow x-y = \frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs} \in \mathbb{Q} \Rightarrow$ The relation is also transitive.

(4) Reflexive: $(s_1,t_1)\,\mu\,(s_1,t_1)$, because both $s_1 \rho s_1$ and $t_1 \sigma t_1$ be the reflexivit of ρ and σ.

Antisymmetric: Let $(s_1,t_1)\,\mu\,(s_2,t_2)$ and $(s_2,t_2)\,\mu\,(s_1,t_1) \Rightarrow s_1 \rho s_2, s_2 \rho s_1, t_1 \sigma t_2, t_2 \sigma t_1 \Rightarrow s_1 = s_2$ and $t_1 = t_2$ by the antisymmetry of ρ and σ.

Transitive: $(s_1,t_1)\,\mu\,(s_2,t_2)$ and $(s_2,t_2)\,\mu\,(s_3,t_3) \Rightarrow s_1 \rho s_2, s_2 \rho s_3, t_1 \sigma t_2, t_2 \sigma t_3 \Rightarrow s_1 \rho s_3$ and $t_1 \sigma t_3$ by the transitivity of ρ and $\sigma \Rightarrow$ $(s_1,t_1)\,\mu\,(s_3,t_3)$.

(5)



$\{0,1,2\}$ is maximal and greatest,

$\emptyset$ is minimal and least.

Discrete Math - Clex B - Solutions

(6) Reflexive: $x+y = x+y$ ✓

Symmetric: $x+y = z+w \Rightarrow z+w = x+y$ ✓

Transitive: $x+y = z+w,\ z+w = t+u \Rightarrow x+y = t+u$ ✓

This is an equivalence relation. The equivalence classes correspond to the elements on $\mathbb{N}$ as follows:

$\{(0,0)\} \to [0]$; $\{(0,1),(1,0)\} \to [1]$; $\{(0,2),(1,1),(2,0)\} \to [2]$ etc.

(7) Reflexive: $\forall a \in A$, $(a,a) \in \rho \wedge (a,a) \in \sigma \Rightarrow (a,a) \in \rho \cap \sigma$

Antisymmetric: Let $(a,b) \in \rho\cap\sigma$ and $(b,a) \in \rho\cap\sigma$. Then $(a,b) \in \rho$ and $(b,a) \in \rho$.
$\Rightarrow b = a$ since ρ is antisymmetric. Thus $\rho\cap\sigma$ is antisymmetric.

Transitivity: Let $(a,b) \in \rho\cap\sigma$ and $(b,c) \in \rho\cap\sigma$. Then these two pairs belong to both ρ and σ. By the transitivity of ρ and σ we have $(a,c) \in \rho$ and $(a,c) \in \sigma \Rightarrow (a,c) \in \rho\cap\sigma$.

(8) $22x + 14 = 8 - 29x$ in $\mathbb{Z}_{37}$

$51x = -6$ in $\mathbb{Z}_{37}$

$14x = 31$ in $\mathbb{Z}_{37}$ $\Rightarrow$ $x = [14]^{-1} 37$ in $\mathbb{Z}_{37}$

To find this reciprocal use Euclidean algorithm.

$37 = 2\cdot 14 - 9$, $14 = 1\cdot 9 + 5$, $9 = 1\cdot 5 + 4$, $5 = 1\cdot 4 + 1$

$1 = 5 - 4 = 5 - (9-5) = 2\times 5 - 9 = 2(14-9) - 9 = 2\times 14 - 3\times 9 =$

$= 2\times 14 - 3(37 - 2\times 14) = -3\times 37 + 8\times 14 \Rightarrow [14]^{-1} = 8$

$x = 8\cdot 31 = 248 = 26$ in $\mathbb{Z}_{37}$.