Discrete Math - Clex 15 - Solutions

(1) 

(2) a) Each of the n vertices is connected to the remaining $n-1$ vertices by an arc : $n(n-1)/2$ arcs.

b) This is clear by a) since a complete graph has the maximum number of arcs.

c) Yes. K_5 has 5 vertices and $5 \cdot 4/2 = 10$ arcs.

(3) a) DNE. Connected graph with 6 nodes and 5 arcs is a tree, thus it cannot have a cycle.

b) DNE. Total degree $= 12 \Rightarrow 6$ arcs. Since it has 6 nodes, this graph cannot be a forest.

c) Yes :

d) DNE. Such a graph has at most 7 internal nodes.

(4) By induction on the number of arcs a. Base case $a=0$. Then $n=1$, $r=1$ and $1-0+1=2$. ✓

Assume the formula holds for the planar representation of any connected plannar graph with k arcs. Consider a graph G with $k+1$ arcs.

Case 1: G has a node of degree 1:  . Erasing this node and the arc connected to it leaves a graph with k arcs for which $n-a+r=2$. For G $(n+1)-(a+1)+r=2$ is therefore true.

Case 2 : G has no node of degree 1. Then we erase an arc like this  . This results in a graph with n arcs for which $n-a+r=2$. G has one more and separates one more region, thus $n-(a+1)+(r+1)=2$ holds for G.

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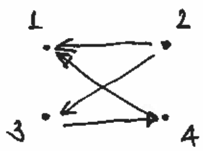
5. Because G is connected there is a path between the vertices u and v. Adding an extra arc between these two vertices results in a cycle from u to u. Adding the arc a created precisely one cycle since if more than one cycle was created we can remove the new arc and a cycle will remain but we started with a tree which has no cycles.

6. a)



; $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ b) $A^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $A^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$A^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $R = A \vee A^{(2)} \vee A^{(3)} \vee A^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$\rho^{\text{trans.cl.}} = \{(2,1),(2,3),(2,4),(3,1),(3,4),(4,1)\}$

7. By induction on the number of arcs. Base case: for a connected graph with $m=0$ arcs there is precisely $m+1 = 0+1 = 1$ vertex.
Assume that the statement is true for graphs with k arcs. Consider a simple graph with $k+1$ arcs, G.

Case 1: G has a node of degree 1: Erasing this node and the adjacent arc leaves a graph with k arcs for which $|\text{nodes}| \leq k+1$. For G we have $|\text{nodes}| \leq (k+1)+1 = k+2$

Case 2: G has a cycle. Erasing an arc from this cycle leaves a graph with k arcs for which we have $|\text{nodes}| \leq k+1$. G has the same number of nodes so we have: $|\text{nodes}| \leq k+1 < k+2$.