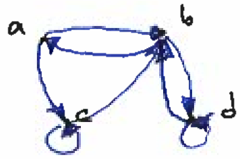
Discrete Math - Clex 16 - Solutions

1) The graph on the left has neither Euler path nor Hamiltonian cycle. The graph on the right has an Euler path: a-b-c-f-e-c-g-b-d-g-e-d-a and a Hamiltonian cycle: a-b-c-f-e-g-d-a.

2) 

$$A^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = R$$

3) i) Let A be the adjacency matrix of G, $A = \{a_{ij}\}$. Then $A^2[i,j] = \sum_{k=1}^{n} a_{ik} a_{kj}$. If $a_{ik} = p$ and $a_{kj} = q$ then there are (pq) paths of length 2 from i to j going through k. Adding the mutually exclusive possibilities for the paths of length 2 to use any node on the graph as an intermediary point, we have $A^2[i,j] = \sum_{k=1}^{n} a_{ik} a_{kj}$ paths of length 2 from node i to node j.

ii) By induction. The base case was settled in i). Assume $A^k[i,j]$ counts the number of paths of length k from i to j. Then

$$A^{k+1}[i,j] = \sum_{\ell=1}^{n} A^k[i,\ell]\, a_{\ell j}$$

clearly counts the possibilities for paths of length $k+1$ from i to j similarly to the argument from part i) in the proof.

4) A connected directed graph will have an Euler path iff either:

a) All nodes have out-degree equal to their in-degree.

b) One node has an in-degree greater by 1 than its out-degree, and another node has an out-degree greater by 1 than its in-degree, while all other nodes have equal in- and out-degrees.

5) This graph has no leafs and no nodes with more than two edges adjacent to them; it is a polygon. A polygon has a Hamiltonian cycle; start at any node.

6) i) Level 1 : a ; Level 2 : b, d ; Level 3 : c, g, e ; Level 4 : f

ii) a - b - c - e - d - (e) - f - (e) - g

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7) i)

$D=\{1\}$

	1	2	3	4	5	6	7	8
d	0	5	∞	5	∞	∞	∞	∞
p	-	1	1	1	1	1	1	1

; $D=\{1,4\}$

	1	2	3	4	5	6	7	8
d	0	5	∞	5	11	∞	∞	∞
p	-	1	1	1	4	1	1	1

$D=\{1,2,4\}$

	1	2	3	4	5	6	7	8
d	0	5	9	5	11	∞	6	∞
p	-	1	2	1	4	1	2	1

; $D=\{1,2,4,7\}$

	1	2	3	4	5	6	7	8
p	0	5	9	5	7	∞	6	∞
d	-	1	2	1	7	1	2	1

$D=\{1,2,4,5,7\}$

	1	2	3	4	5	6	7	8
d	0	5	9	5	7	10	6	∞
p	-	1	2	1	7	5	2	1

: $D=\{1,2,3,4,5,7\}$

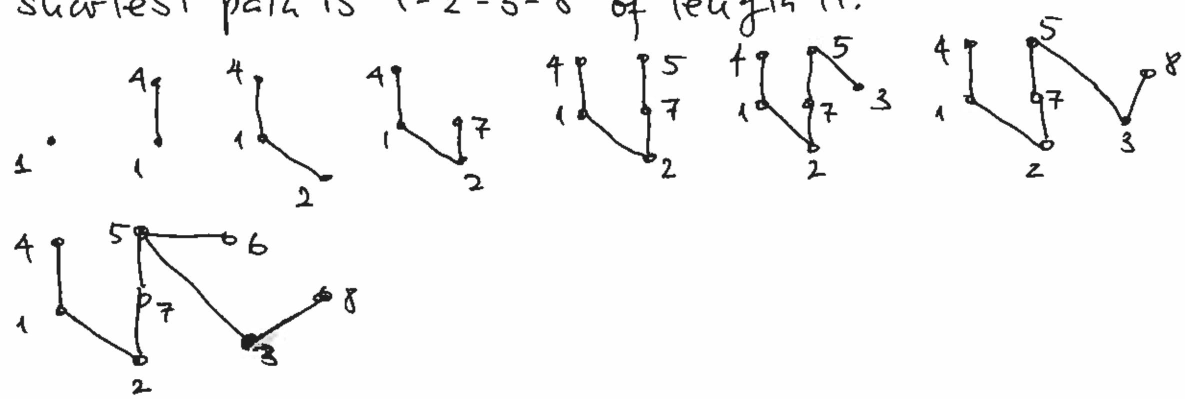
	1	2	3	4	5	6	7	8
p	0	5	9	5	7	10	6	11
d	-	1	2	1	7	5	2	3

$D=\{1,2,3,4,5,6,7\}$

	1	2	3	4	5	6	7	8
d	0	5	9	5	7	10	6	11
p	-	1	2	1	7	5	2	3

The shortest path is 1-2-3-8 of length 11.

ii)



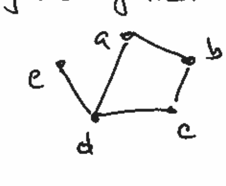
8) a) Euler paths exist

- for $m=n=1$
- for $m=2$ and $n\in\mathbb{N}_{odd}$ or vice versa
- for $m,n\in\mathbb{N}_{even}$

b) Hamiltonian circuits exist for $m=n\geq 2$.

9) By contradiction. Suppose that T is a minimal spanning tree that does not include the arc a. Arc a, is then part of a cycle. Remove a different higher weight arc from this cycle. This result in a spanning tree with lower weight.

10) As the depth-first search algorithm is moving through the graph if the current node has on its adjacency list a visited node other than the parent node the graph has a cycle.



The already visited node a is on the adjacency list of node d.