Disc Math - H16 - Clex 8 - Solutions

1 a) $C(k) = 3C(k-1) + 1 = 3\left(3C(k-2)+1\right)+1 = 3^2 C(k-2) + 3 + 1 =$

$= 3^2\left(3C(k-3)+1\right)+3+1 = 3^3 C(k-3) + 3^2 + 3 + 1 = \ldots =$

$= 3^{k-1}\underbrace{C(1)}_{1} + 3^{k-2} + 3^{k-3} + \ldots + 3^2 + 3 + 1 = \sum_{i=1}^{k} 3^{i-1} = \sum_{i=0}^{k-1} 3^i =$

$= \frac{3^k-1}{3-1} = \frac{3^k-1}{2}$

b) Base case $C(1) = \frac{3^1-1}{2} = 1$. Assume $C(k) = \frac{3^k-1}{2}$. Need t

prove $C(k+1) = \frac{3^{k+1}-1}{2}$

$C(k+1) = 3C(k) + 1 = 3\cdot\frac{3^k-1}{2} = \frac{3^{k+1}-3+2}{2} = \frac{3^{k+1}-1}{2}$. ✓

2 $C(1) = 1$, $C(2) = 4$, $C(k) = 3\left(C(k-1) - C(k-2)\right) + C(k-1) =$

$= 4C(k-1) - 3C(k-2)$

$t^2 - 4t + 3 = 0$; $t_1 = 1$, $t_2 = 3$; $C(n) = p\,3^{n+1} + q$.

Initial conditions $\Rightarrow$ $p = 3/2$, $q = 1/2$ $\Rightarrow$ $C(n) = \frac{3^n-1}{2}$

a) $C(20) = \frac{3^{20}-1}{2} = 1.743 \times 10^9 = 1.743$ billion

b) $C(k) = \frac{3^k-1}{2} = 7\times 10^9$, $k = \frac{\ln(14\times 10^9+1)}{\ln 3} = 21.26$, $k = 22$

3 $r^2 - 6r + 5 = 0$, $r_1 = 5$, $r_2 = 1$, $F(n) = p\,5^{n-1} + q$

$F(1) = p + q = 8$, $F(2) = 5p + q = 16$ $\Rightarrow$ $p = 2$, $q = 6$, $F(n) = 2\cdot 5^{n-1} + 6$

4 $r^2 - 4r + 4 = 0$, $r_{1,2} = 2$, repeated root

$C(n) = p\cdot 2^{n-1} + q(n-1)2^{n-1}$; $-2 = C(1) = p$; $12 = C(2) = 2p + 2q$

$\Rightarrow p = -2$, $q = 8$, $C(n) = -2\cdot 2^{n-1} + 8(n-1)2^{n-1} = n2^{n+2} - 5\cdot 2^n$

5 $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{4x^2+3x+5\ln x}{x^2} \stackrel{L'H.}{=} \lim_{x\to\infty}\frac{8x+3+5/x}{2x} =$

$= \lim \left(4 + \frac{3}{2x} + \frac{5}{2x^2}\right) = 4 \Rightarrow f = \Theta(g)$

6. $\log(3x^2) = \log 3 + 2\log x \leq 3\log x$ for $x \geq 3$

$\log(x) \leq \frac{1}{2}(\log(3x^2)) = \frac{1}{2}\log 3 + \log(x)$

since each function asymptotically bounds the other they have the same order of magnitude.

7. $\lim_{x\to\infty} \frac{(\ln x)^2}{x^{1/2}} = \lim_{x\to\infty} \frac{2\ln x \cdot 1/x}{1/2\sqrt{x}} = 4 \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} =$

$= 4 \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = 8 \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$