DiscMath - §1 - Logical Statements

→ mechanize reasoning; strip way confusing, ambiguous verbiage

Def: A statement (proposition, assertion) is a sentence which is either true or false.

Ex: $2>0$ is a statement; $-3<-5$ is a statement; $x>3$ is not a statement.

My neighbour Gaetan is an electrician. (statement)

She is a truck driver. (Not a statement).

Compound statements: (statements are combined with logical connectives)

Def: $\wedge$ = and, $p \wedge q$ conjunction (p, q - statements)

$\vee$ = or, $p \vee q$ disjunction

$' = \sim = \neg$, $p' = \sim p = \neg q$ negation.

Ex: $(2 \in \mathbb{N}) \wedge (2 \in \mathbb{R}^+)$; $\sim(-2 \in \mathbb{N})$; $(-2 \in \mathbb{N}) \vee (-2 \in \mathbb{Z})$.

Ex: $p = \{3<2\}, q = \{3=2\} \Rightarrow p \vee q = \{3 \leq 2\}$, $\sim p = \{3 \geq 2\}$

Ex: h = It is hot ; s = It is sunny ; m = It is muggy

a) It is sunny but not hot $\quad s \wedge \sim h$

b) It is neither hot nor muggy $\quad \sim h \wedge \sim m$

c) It is sunny or hot and muggy

Rules: $\sim$ binds expressions stronger than $\wedge, \vee$

$\wedge, \vee$ bind equally.

$\sim p \vee q$, $p \vee (q \wedge r)$ are well formed; $p \vee q \wedge r$ isn't.

The truth values of the elementary staments entail truth values for the compound statements:

p	$\sim p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

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rem: There is also the exclusive or: xor, $\oplus$, ...

p	q	p xor q
T	T	F
T	F	T
F	T	T
F	F	F

ex: Form the truth table of $\sim((p\wedge q)\vee \sim r)$.

p	q	r	$p\wedge q$	$\sim r$	$(p\wedge q)\vee\sim r$	$\sim((p\wedge q)\vee \sim r)$
T	T	T	T	F	T	F
T	T	F	T	T	T	F
T	F	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	F	F	F	T
F	T	F	F	T	T	F
F	F	T	F	F	F	T
F	F	F	F	T	T	F

Logical equivalence:

$(p\wedge q) \equiv (q \wedge p)$

ex:

p	q	$p\wedge q$	$q\wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

It is hot and muggy
It is muggy and hot.

ex: De Morgan's laws: $\sim(p\wedge q) \equiv \sim p \vee \sim q$ $\quad\sim(p\vee q)\equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$p\wedge q$	$\sim(p\wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The second one is similar.

ex: $\sim\{-1 \geq x \vee x > 4\} \equiv$
$= \{-1 < x \wedge x \leq 4\} \equiv \{-1 < x \leq 4\}$.

Conditionals: If p then q. $\quad$ p - antecedent ; p - consequent

p	q	$p\Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

ex: If I study well I will do well on the test.

rem: A conditional statement that is true because the hypothesis is false is "true by default"!

ex: If $0=1$ then $55=4$ is True!

rem: $p \to q \equiv \sim p \vee q$

p	q	$\sim p$	$\sim p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Notation: $\to$ binds weaker than any of the connectives introduced so far

$$p \vee \sim q \to \sim p = (p \vee (\sim q)) \to (\sim p).$$

rem: The last connective to be applied is the main connective. Above the main connective is $\to$.

rem: If .. then .. statements are frequently phrased in terms of necessary / sufficient condition.

$p \to q \equiv q$ is a necessary condition for p; if p is true q is true as well

$\equiv p$ is a sufficient condition for q

ex: Rewrite: "To enter the club a person must be at least 18 years old".

If a person can enter the club then she is at least 18.

Being 18 is a necessary condition for a person to enter the club.

Entering the club is a sufficient condition for a person to be at least 18

Def: The contrapositive of $p \to q$ is $\sim q \to \sim p$

Th: A conditional statement is logically equivalent to its contrapositive

p	q	$\sim q \to \sim p$	$p \to q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$\to$ This is one of the most useful logical equivalences.

ex: If Rejean lives in Laval, then he lives in Quebec $\overset{c.p.}{\to}$ If Rejean does not live in Quebec, then he does not live in Laval.

rem: For a conditional $p \to q$, the converse is $q \to p$ and the inverse is $\sim p \to \sim q$

The converse and the inverse are equivalent to each other but not to the original

Def: The biconditional connective $\leftrightarrow$ (equivalence connective) is defined as

p	q	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

and as the extended table shows

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

rem: $p \leftrightarrow q$ is (always) true iff $p \equiv q$.

rem: $\leftrightarrow$ binds the weakest.

ex: $(p \to q) \leftrightarrow (q' \to p')$ is true for any truth values of p and q.

Def: A (compound) logical statement whose truth values are always true (regardless of the truth values of the variables) is called a tautology.

ex: $(p \to q) \leftrightarrow (q' \to p')$ and $p \vee p'$ are tautologies.

Def: A logical statement whose truth values are always false is called a contradiction.

ex: $p \wedge p'$ is a contradiction.

rem: $p \leftrightarrow q$ is a tautology iff $p \equiv q$.

not: A tautology will be represented by 1; a contradiction by 0.

ex: $A \vee 0 \equiv A$, $A \wedge 0 = 0$; $A \wedge 1 = A$, $A \vee 1 = 1$

ex: $(A \vee B) \vee C \equiv A \vee (B \vee C)$ then simply written as $A \vee B \vee C$.

$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C) \equiv A \wedge B \wedge C$.

ex: Are $(A \to B) \to C$ and $A \to (B \to C)$ logically equivalent?

A	B	C	$A \to B$	$B \to C$	$(A \to B) \to C$	$A \to (B \to C)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Not equivalent!

HW: §1.1: p16
6, 11, 16, 21, 24, 29, 30, 36, 40, 47, 61