Disc Math - §11 - Sets

"Def": A **set** is a collection of objects.

def: $a \in A$ means a is an element of the set A.

ex: $5.72 \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$

def: Two sets are equal if they have the same elements.

ex: $A = \{2, a, b\}$ $\quad B = \{2, a, a, b, 2, a\}$ $\quad A = B$

def: A set A is **finite** if it has n elements, $n \in \mathbb{N} \cup 0$; otherwise A is infinite.

ex: $A = \{1, 2, 3, L\}$ - finite ; $\mathbb{N}, \mathbb{R}$ - infinite

$\emptyset$ - empty set (finite).

constr: Sets are frequently described as truth sets of predicates

$A = \{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\} = \{1, 2\}$

$B = \{n \in \mathbb{N} \mid 3 \mid n\} = \{3, 6, 9, \ldots\}$

$C = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ - upper half Cartesian plane.

Def: A set A is a **subset** of a set B, $A \subseteq B$ if $\forall x, x \in A \to x \in B$.

If $A \subseteq B \wedge A \neq B$, A is a proper subset of B, $A \subset B$.

ex: $\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R}$; $\emptyset \subset \mathbb{N}$, $\emptyset \subseteq \emptyset$

ex: Which of the following relations is true? $A = \{1, a, \emptyset\}$, $B = \{\{1\}, 2\}$

$1 \in A$, $\emptyset \in A$, $\emptyset \in B$, $\emptyset \subset B$, $1 \in B$, $a \subset A$, $\{2\} \subset B$, $\{1\} \subset B$?

rem: $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

Def: For a set S the set of subsets of S, denoted $P(S)$ is the **powerset** of S.

ex: $A = \{a, b, c\}$ $\quad P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$

Ex: $P(\emptyset) = \{\emptyset\}$.

Stud. Ex: $P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

Prop: For a set S, with $\#S$, $\#P(S) = 2^n$.

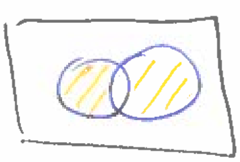
Pr: Induction: $\#\emptyset = 0$, $\#P(\emptyset) = 1 = 2^0$.

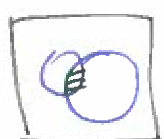
Assume for any set with k elements, the power set has 2^k elements. Consider a set S, $\#S = k+1$. Split S into a set T with $\#T$ and a single element x. The subsets of T are subsets of S and there are 2^k of them. To each of these we can add x; that is another 2^k subsets of S $\Rightarrow$ $\#P(S) = 2^k + 2^k = 2^{k+1}$! ∎

rem: Another way to see that $\#S = n \Rightarrow \#P(S) = 2^n$ is this: every element of S is either in the subset or not $\to \{T, F\}$. There are $2 \times 2 \times \cdots \times 2 = 2^n$ choices.

context: Typically we operate on subsets of some largest set U $\to$ universe of discourse (frequently $\mathbb{R}$).

Def: Let $A, B \in P(U)$. Their union, denoted by $A \cup B$, is $\{x \mid x \in A \lor x \in B\}$. Their intersection, denoted by $A \cap B$, is $\{x \mid x \in A \land x \in B\}$.


$A \cup B$;

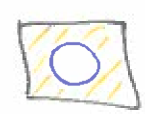

$A \cap B$

A, B are disjoint if $A \cap B = \emptyset$.

Ex: Prove that $A \cap B \subseteq A \cup B$.

$x \in A \cap B \to \{x \in A \land x \in B\} \Rightarrow \{x \in A\} \to \{x \in A \lor x \in B\} \Rightarrow x \in A \cup B$.

Def. For $A \in P(U)$, the complement of A, A' is $\{x \in U \mid x \notin A\}$.



Ex: $U = \mathbb{R}$, $A = \mathbb{Q}$ $\to$ A' = irrationals.

Def: For $A, B \in P(U)$ their set difference $A \setminus B = A - B$ is $\{x \mid x \in A \land x \notin B\}$

Ex: $\mathbb{Z} \setminus \mathbb{Z}_{odd} = \mathbb{Z}_{even}$

Ex: $A = \{1,2,3,5,10\}$ $B = \{2,4,7,8,9\}$ $C = \{5,8,10\}$ $U = \{1,\dots,10\}$

$A \setminus C = \{1,2,3\}$; $B' \cap (A \cup C) = \{1,3,5,10\}$

Def: Let $A, B \in P(U)$. The cartesian product of A and B, denoted by $A \times B$ is: $A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$.

Ex: $A = \{1,2\}$ $B = \{a,2\}$ $A \times B = \{(1,a),(1,2),(2,a),(2,2)\}$.

$A \times \phi = \phi$, $\forall A$. $B \times A = ?$

Prop: $\# A \times B = \# A \cdot \# B$.

rem: Instead of $A \times A$ we write A^2. Similarly $A^n = A \times \dots \times A$.

Ex: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2, \dots, \mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$; $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ etc.

Ex: True or false: $P(A \times B) = P(A) \times P(B)$

sets of ordered pair   ordered pairs of subsets

$\# P(A \times B) = 2^{mn} \neq \# P(A) \cdot \# P(B) = 2^m \cdot 2^n = 2^{m+n}$

Ex: True or false: $P(A \cap B) = P(A) \cap P(B)$

element here is a subset of $A \cap B$ element here is a set which is a subset of both A and B, i.e. a subset of $A \cap B$.

Set Identities → Boolean Algebra.

① $A \cup B = B \cup A$ ①' $A \cap B = B \cap A$ "commutativity"

② $(A \cup B) \cup C = A \cup (B \cup C)$ ②' $(A \cap B) \cap C = A \cap (B \cap C)$ "associativity"

③ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ③' $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ "distributivity"

④ $A \cup \phi = A$ "identity" ④' $A \cap U = A$ ⑤ $A \cup A' = U$ ⑤' $A \cap A' = \phi$ "complement"

⑤ $(A \cup B)' = A' \cap B'$ ⑤' $(A \cap B)' = A' \cup B'$

ex: Here is a proof of $(A\cap B)' = A'\cup B'$ from the definitions:
Let $x\in(A\cap B)'$. Then $x\notin A\cap B$, and since $A\cap B=\{y \mid y\in A \wedge y\in B\}$, either $x\notin A$ or $x\notin B$. If $x\notin A$, then $x\in A'\subseteq A'\cup B'$; otherwise, $x\in B$ so $x\in B'\subseteq A'\cup B'$. Thus $x\in A'\cup B'$ and hence $(A\cap B)'\subseteq A'\cup B'$.

For the opposite inclusion we will prove it by contradiction.
Suppose $\exists x\in A'\cup B'$ s.t. $x\notin(A\cap B)'$. Then $x\in A\cap B \to \{x\in A \wedge x\in B\}$
But then $x\notin A' \wedge x\notin B' \to x\notin A'\cup B'$. Contradiction. $\to A'\cup B'\subseteq(A\cap B)'$
Thus $A'\cup B' = (A\cap B)'$.

rem: From these basis set identities we can prove many more.

ex: Prove the identity $[C\cap(A\cup B)]\cup[C'\cap(A\cup B)] = A\cup B$

$$[C\cap(A\cup B)]\cup[C'\cap(A\cup B)] \underset{\text{distr.}}{=} (C\cup C')\cap(A\cup B) \underset{\text{comp.}}{=} U\cap(A\cup B)$$
$$\underset{\text{ident.}}{=} A\cup B$$

rem: Since the basic laws are symmetric under duality $\cap \Leftrightarrow \cup$, $U\Leftrightarrow\emptyset$ any identity we prove has a dual version.

ex Contd: From the above example we also have:
$$[C\cup(A\cap B)]\cap[C'\cup(A\cap B)] = A\cap B.$$

ex: Prove $A\cap\left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n}(A\cap B_i)$

Pr: By induction. Base case $n=1$, trivial, $A\cap B_1 = A\cap B_1$.
Assume the identity holds for $n=k$, have to prove it for $n=k+1$
$$A\cap\left(\bigcup_{i=1}^{k+1} B_i\right) = A\cap\left(\bigcup_{i=1}^{k} B_i \cup B_{k+1}\right) = \left(A\cap\bigcup_{i=1}^{k} B_i\right)\cup(A\cap B_{k+1}) =$$
$$=\left(\bigcup_{i=1}^{k} A\cap B_i\right)\cup(A\cap B_{k+1}) = \bigcup_{i=1}^{k+1} A\cap B_i.$$

Algebra: sets with operations.
ex: $\mathbb{Z}$ comes with the binary operations $+, -, \cdot$

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Ex: $\mathbb{N}$ comes with the binary operations $+, \cdot$, but is not closed under $-$.

Ex: $P(M)$ comes with the binary operations $\cap, \cup$ (and $\setminus$) and the unary operation $'$. These obey a number of basic identities (axioms) $\mapsto$ Boolean algebra.

Russel's Paradox

Ex: A set which contains itself as an element $A = \{1, 2, 3, A\}$

Ex: The set of mathematical abstractions.

Russell's Paradox: Let S be the set of all sets which are not elements of themselves.

$$S = \{A \mid A \text{ is a set and } A \notin A\}.$$

Is S an element of itself?

→ If $S \in S$ then S is a set which is not a member of itself,
$\Rightarrow S \notin S$

→ If $S \notin S$, then S is a set which is a member of itself, $S \in S$

rem: There is a male barber who shaves all those men, who do not shave themselves. Does the barder shave himself?

⇝ One possible resolution of Russel's paradox: S is a class.

rem: Berry paradox: "Consider, the smallest positive integer not describable in fewer than 12 English words"

HW § 4.1 p 239

14, 24, 32, 53, 66 (c), 74, 83 (a, c), 93, 101, 103, 105