Disc Math - §12 - Cardinality

Def: Two sets A and B have the same cardinality if there is 1-1 function from A to B.

Ex: $\mathbb{Z}$ and $\mathbb{Z}_{even}$ have the same cardinality.

$$f:\mathbb{Z}\to\mathbb{Z}_{even},\quad f(n)=2n.$$

rem: An infinite set has the same cardinality as a proper subset of itself.

Def: A set is called countably infinite if it has the same cardinality as $\mathbb{N}$. A set is countable if it is finite or countably infinite. A set that is not countable is called uncountable.

Ex: $\mathbb{Z}$ is countable.

Pr: Here is a bijection $f:\mathbb{N}\to\mathbb{Z}$, $f(n)=\begin{cases} n/2 & n\in\mathbb{N} \text{ even} \\ -(n-1)/2 & n\in\mathbb{N} \text{ odd.}\end{cases}$

Th: $\mathbb{Q}$ is countable (i.e. $\mathbb{Q}$ has the same cardinality as $\mathbb{N}$).

Pr: We prove first that $\mathbb{Q}^+$ is countable. Arrange $\mathbb{Q}^+$ on a grid:

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

$f:\mathbb{N}\to\mathbb{Q}^+$

$f(1)=1$, $f(2)=\frac{1}{2}$, $f(3)=2$, $f(4)=3$, $f(5)=\frac{1}{3}$, ...

$\to$ $\forall$ positive rational is somewhere on the grid (the function is onto).

$\to$ By skipping the numbers already encountered the function is 1-1.

Thus $\mathbb{Q}^+\sim\mathbb{N}$. Now observe $\mathbb{Q}=\{\mathbb{Q}^-\cup\{0\}\cup\mathbb{Q}^+\}\sim\{\mathbb{N}\cup\{0\}\cup\mathbb{N}\}\sim\mathbb{Z}\sim\mathbb{N}$.

Cantor diagolization process.

rem: $\forall x\in(0,1)$ could be written in the form $0,a_1a_2a_3\ldots$, $a_i\in\{0,\ldots,9\}$.

However $0.2999\ldots = 0.30\ldots$ So assume $\forall$ number ending in $\ldots 999\ldots$ is replaced by the corresponding number ending in $\ldots 000\ldots$

Th: (Cantor). The set of real numbers in the interval $(0,1)$ is uncountable.

Pr: By contradiction. Suppose the set $(0,1)$ is countable. List the numbers

$0.\,a_{11}\,a_{12}\,a_{13}\,\ldots\,a_{1n}\ldots$

$0.\,a_{21}\,a_{22}\,a_{23}\,\ldots\,a_{2n}\ldots$

$\ldots$

$0.\,a_{n1}\,a_{n2}\,a_{n3}\,\ldots\,a_{nn}\ldots$

$\ldots$

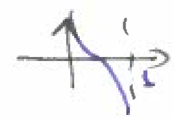
Construct a new decimal number $d = 0.d_1 d_2 d_3 \ldots$ as follows:

$$d_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1. \end{cases}$$

Thus $\forall n \in \mathbb{N}$, d differs in the n'th decimal position from the n'th number on the list $\Rightarrow$ d is not on the list. (Adding d to the list, wouldn't help, since we can construct another counterexample).

Cor: The set of real numbers $\mathbb{R}$ is uncountable.

Pr: $\cot(\pi x) : (0,1) \to \mathbb{R}$ is a bijection:



A bit of cardinality arithmetic.

Th: Any subset of a countable set is countable.

Pr: Let A be an arbitrary countable set and let $B \subseteq A$. If B is finite we are done. Suppose B is infinite. Since A is countable its elements could be arranged in a sequence: $a_1, a_2, a_3, \ldots$ Define $g: B \to \mathbb{N}$ as follows: $g(i)$ is the i'th element of B in the list for A. g is clearly a bijection. ☒

Ex: The set of primes is countable.

Th: (Cor). Any set with uncountable subset is uncountable.

Pr: This is just the contrapositive of the previous Theorem. ☒

Ex: Any nonempty interval in $\mathbb{R}$ is uncountable.

Th: (Cantor). For $\forall$ set A, A and $P(A)$ have different cardinalities.

Pr: Let a function $f: A \to P(A)$ be given. It cannot be onto. Indeed define

$$B = \{x \in A : x \notin f(x)\}.$$ Then $B \in P(A)$

But then if $\exists a \in A$, $f(a) = B$, then:

- if $a \in B$, then by the definition of B, $a \notin f(a) = B$, contradiction.
- if $a \notin B$, then $a \notin f(a) \Rightarrow a \in B$, contradiction.

$\Rightarrow \nexists a \in A$, $f(a) = B \Rightarrow \nexists$ bijection $f : A \to \mathcal{P}(A) \Rightarrow \mathrm{card}(A) < \mathrm{card}(\mathcal{P}(A))$ $\boxtimes$

ex: $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ (the set of functions from $\mathbb{N}$ to $2 = \{0,1\}$). But then $2^{\mathbb{N}} = [0,1]$ via the map $(a_1, a_2, a_3, \dots) \to \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$. Thus $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}} \simeq [0,1] \simeq \mathbb{R}$.

Axiom: (continuum hypothesis) There is no set whose cardinality is strictly between that of $\mathbb{N}$ and that of $\mathbb{R}$.

$\nexists S$ s.t. $\aleph_0 < \mathrm{card}(S) < 2^{\aleph_0}$ or equivalently $2^{\aleph_0} = \aleph_1$.